CH 1 - P1

Solution of nonlinear equations

$$p(x) = ax^2 + bx + c$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$p(x_1) = p(x_2) = 0.$$

No such formula if p is a polynomial of degree $\geqslant 5$.

Similarly for the roots of many nonlinear functions.

Root-finding problem (RFP)

$f : \mathbb{R} \to \mathbb{R}$ continuous on $[a, b]$

Find $x \in [a,b]$ s.t. $f(x) = 0$.

THM

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ such that $f(a) \cdot f(b) \leqslant 0$.

Then there exists $x \in [a, b]$ such that $f(x) = 0$.

Proof

If $f(a)=0$ or $f(b)=0$, nothing to prove.
Suppose

$$f(a)\cdot f(b) < 0 \implies \begin{array}{c} f(a) < 0, \; f(b) > 0 \\ \text{or} \\ f(a) > 0, \; f(b) < 0. \end{array}$$

In either case, $f(x) = 0$ $\exists x \in [a,b]$ by IVT. $\square$

Ex

$$f(x) = e^x - x - 2$$

$$f(1) = e-3 < 0$$
$$f(2) = e^2-4 > 0$$

Hence, f has a root on $[1,2]$.

Fixed-point problem (FPP)

$g: \mathbb{R} \to \mathbb{R}$ continuous on $[a,b]$

Find $x \in [a,b]$ s.t. $g(x) = x$

RFP and FPP are related

$$f(x) = 0 \iff x = \underbrace{f(x) + x}_{g(x)}$$

THM (Brouwer's Fixed Point Thm)

Let $g: \mathbb{R} \to \mathbb{R}$ be continuous on $[a,b]$ such that $g(x) \in [a,b]$ for all $x \in [a,b]$.

Then there exists $x \in [a,b]$ such that $g(x) = x$.

Ex

$$f(x) = e^x - x - 2$$
$$g(x) = \ln(x+2)$$

(i.e. $e^x - x - 2 = 0$
$\Leftrightarrow e^x = x+2$
$\Leftrightarrow x = \ln(x+2)$)

$$g(1) = \ln 3 > 1$$
$$g(2) = \ln 4 < 2$$

additionally
$g'(x) = 1/(x+2) > 0$
$\forall x \in [1,2]$
i.e., g is monotone increasing on $[1,2]$.

Hence, $g(x) \in [\ln 3, \ln 4] \subseteq [1,2]$
$\forall x \in [1,2]$

By Brouwer's fixed point thm, $g(x) = x$ $\exists x \in [1,2]$.

Proof

Let $f(x) = x - g(x)$, observe

$$f(a) = a - g(a) \leq 0 \quad \text{and} \quad f(b) = b - g(b) \geq 0$$
$$\Longrightarrow f(a) \cdot f(b) \leq 0$$
$$\xRightarrow{\text{(by previous thm)}} f(x) = 0 \quad \exists x \in [a,b]$$
$$\Longrightarrow g(x) = x \quad \exists x \in [a,b]. \qquad \square$$

Ex 2

$f(x) = e^x - x - 2$

$h(x) = e^x - 2$

$h(2) = e^2 - 2 > 2$

BFP is inconclusive regarding whether $h(x) = x$ $\exists x \in [1,2]$.

Simple Iteration

Sup. $g(x) \in [a,b]$ for all $x \in [a,b]$.

(x_k) s.t. $x_{k+1} = g(x_k)$ $\left(\begin{array}{l}\text{for a given} \\ x_0 \in [a,b]\end{array}\right)$

Suppose (x_k) is convergent with $x_* := \lim_{k\to\infty} x_k$. Then

$$x_* = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} g(x_k)$$
$$= g\left(\lim_{k\to\infty} x_k\right) = g(x_*)$$

i.e. x_* is a fixed-point of g.

Ex

$f(x) = e^x - x - 2$

$g(x) = \ln(x+2)$

$x_{k+1} = \ln(x_k + 2)$

$\lim_{k\to\infty} x_k = x_*$

$\Longrightarrow$

$e^{x_*} - x_* - 2 = 0.$

THM (Contraction Mapping)

Suppose $g(x) \in [a,b]$ for all $x \in [a,b]$. Moreover, g is Lipschitz continuous with a Lipschitz constant $L < 1$ on $[a,b]$, that is

$$|g(x) - g(y)| \leqslant L|x-y| \quad \forall x,y \in [a,b].$$

The function g has a unique fixed point on $[a,b]$. Furthermore, the sequence (x_k), $x_{k+1} = g(x_k)$ converges to this unique fixed point.

Proof

Let $x_1, x_2 \in [a,b]$ be fixed points.

$$|x_1 - x_2| = |g(x_1) - g(x_2)| \leqslant L|x_1 - x_2|$$

$$\Longrightarrow (1-L)|x_1 - x_2| \leqslant 0.$$

Since $L < 1$, we have $x_1 = x_2$.

As for the convergence of (x_k), let x_* be the unique fixed point on $[a,b]$.

$$|x_{k+1} - x_*| = |g(x_k) - g(x_*)| \leqslant L|x_k - x_*|.$$

By induction

$$|x_k - x_*| \leqslant L^k |x_0 - x_*|$$

implying

$$\lim_{k\to\infty} |x_k - x_*| = 0 \Longrightarrow \lim_{k\to\infty} x_k = x_*.$$

$\square$

Ex

$f(x) = e^x - x - 2$

$g(x) = \ln(x+2) \qquad x_{k+1} = \ln(x_k + 2)$

Observe $\quad g'(x) = 1/(x+2)$

$$\implies |g'(x)| \leqslant \left[\tfrac{1}{4}, \tfrac{1}{3}\right] \quad \forall x \in [1,2].$$

Hence, by MVT for all $x, y \in [1,2]$

$$\frac{g(x) - g(y)}{x - y} = g'(\varepsilon) \qquad \exists\, \varepsilon \in (1,2)$$

$$\implies |g(x) - g(y)| = |g'(\varepsilon)|\,|(x-y)|$$

$$\implies |g(x) - g(y)| \leqslant \tfrac{1}{3}\,|x - y|$$

Conclusions,

* $g(x) = \ln(x+2)$ has a unique fixed point on $[1,2]$
* (x_k), $x_{k+1} = \ln(x_k + 2)$ converges to this fixed point.

Remark

If $g : \mathbb{R} \to \mathbb{R}$ is differentiable on (a,b) s.t.
$|g'(x)| \leqslant L \quad \forall x \in (a,b)$ for some $L < 1$, then

$$|g(x) - g(y)| \leqslant L\,|x - y| \quad \forall x, y \in [a,b].$$

Ex 2

$$f(x) = x^2 - 2$$
$$g(x) = \frac{2-x^2}{4} + x$$
$$x_{k+1} = \frac{2-x_k^2}{4} + x_k \quad \text{on } [1,2]$$

$$\left(\begin{array}{l} x^2-2=0 \\ \Longleftrightarrow \frac{2-x^2}{4} = 0 \\ \Longleftrightarrow \frac{2-x^2}{4} + x = x \end{array} \right)$$

$g(1) = 5/4 \qquad g(2) = 3/2$

and $g'(x) = -x/2 + 1 \in [0, 1/2] \quad \forall x \in [1,2]$.

Hence, $g(x) \in [5/4, 3/2] \subseteq [1, 2] \quad \forall x \in [1,2]$,

$$|g'(x)| \leqslant 1/2 \quad \forall x \in [1,2],$$
$$|g(x) - g(y)| \leqslant 1/2\, |x-y| \quad \forall x,y \in [1,2].$$

We conclude

(i) $g(x) = \frac{2-x^2}{4} + x$ has a unique fixed-point on $[1,2]$.

(ii) $x_{k+1} = (2-x_k^2)/4 + x_k$ converges to this unique fixed point.

CH 1 - P2

Rate of convergence

Suppose $\lim_{k\to\infty} x_k = x_*$, $g(x_*) = x_*$,

$g'(x)$ is continuous in an open neighborhood of x_*.

$$|x_{k+1} - x_*| = |g(x_k) - g(x_*)|$$
$$\stackrel{\text{(by MVT)}}{=} |g'(\varepsilon_k)|\,|x_k - x_*|$$

for some ε_k in open interval with end-points x_k, x_*

Hence,

$$\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = \lim_{k\to\infty} |g'(\varepsilon_k)|$$
$$= |g'(\lim_{k\to\infty} \varepsilon_k)|$$
$$= |g'(x_*)|.$$

Ex

$$f(x) = x^2 - 2$$
$$g(x) = \frac{2-x^2}{4} + x$$
$$x_{k+1} = \frac{2-x_k^2}{4} + x_k$$

$$\left(\begin{aligned} g'(x) &= -x/2 + 1 \\ g'(\sqrt{2}) &= -\frac{\sqrt{2}}{2} + 1 \\ &= 1 - 1/\sqrt{2} \end{aligned}\right.$$

If $\lim_{k\to\infty} x_k = \sqrt{2}$, then $\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = 1 - 1/\sqrt{2}$

$$h(x) = \frac{x^2+2}{2x}$$

$$x_{k+1} = \frac{x_k^2+2}{2x_k}$$

$$\left(\begin{array}{l} x^2-2=0 \iff \\ x^2+2 = 2x^2 \iff \\ \frac{x^2+2}{2x} = x \end{array} \right.$$

$$h'(x) = 2x \cdot \frac{1}{2x} - \frac{x^2+2}{2x^2}$$

$$h'(\sqrt{2}) = 1 - 1 = 0$$

$$\lim_{k\to\infty} x_k = \sqrt{2} \implies \lim_{k\to\infty} \frac{|x_{k+1}-x_*|}{|x_k - x_*|} = 0$$

Defn

Suppose $\lim_{k\to\infty} x_k = x_*$, and there exists (ε_k) such that $|x_k - x_*| \leqslant \varepsilon_k$ as well as

$$\lim_{k\to\infty} \frac{|\varepsilon_{k+1}|}{|\varepsilon_k|} = \mu.$$

(1) If $\mu \in (0,1)$, the sequence (x_k) converges to x_* at least linearly.

(2) If $\varepsilon_k = |x_k - x_*|$ $\forall k$, and $\mu \in (0,1)$ the sequence (x_k) converges to x_* linearly.

(3) If $\mu = 0$, the sequence (x_k) converges to x_* superlinearly.

(4) If $\mu = 1$, and $\varepsilon_k = |x_k - x_*|$ $\forall k$ the sequence (x_k) converges to x_* sublinearly.

Stable and unstable fixed-points

THM (Stable Fixed-Points)

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous on $[a,b]$, x_* be a fixed-point of g such that $g'(x)$ is continuous in an open neighborhood of x_* and $|g'(x_*)| < 1$.

There exists an open neighborhood I of x_* such that $\lim_{k\to\infty} x_k = x_*$ for (x_k), $x_{k+1} = g(x_k)$ for all $x_0 \in I$.

Proof

Let L be such that $|g'(x_*)| < L < 1$.

There exists $I = (x_* - \delta, x_* + \delta)$ such that $|g'(x)| \leq L \quad \forall x \in I$. (due to continuity of $g'(x)$ in a neighborhood of x_*).

Suppose $x_k \in I$. Then by MVT $\exists\, \varepsilon_k \in I$

$$|x_{k+1} - x_*| = |g(x_k) - g(x_*)| = |g'(\varepsilon_k)|\,|x_k - x_*|$$
$$\leq L\,|x_k - x_*|$$

$$\Longrightarrow x_{k+1} \in I.$$

It follows by induction that for all $x_0 \in I$,

$$x_k \in I \;\; \forall k \quad \text{and} \quad |x_k - x_*| \leq L^k |x_0 - x_*|$$

implying $\lim_{k\to\infty} x_k = x_*$ as desired. $\square$

THM (Unstable Fixed-Point)

Let g and x_* be as in the previous thm with the only exception $|g'(x_*)| > 1$.

For all x_0 s.t. $x_0 \neq x_*$, we have that the sequence (x_k), $x_{k+1} = g(x_k)$ does not converge to x_*.

Proof

As in the previous proof $\exists L$ s.t. $|g'(x_*)| > L > 1$ and an interval $I = (x_* - \delta, x_* + \delta)$ satisfying $|g(x)| \geqslant L \quad \forall x \in I$.

Suppose $x_k \in I$. But then again MVT implies the existence of an $\xi_k \in I$ such that

$$|x_{k+1} - x_*| = |g'(\xi_k)|\,|x_k - x_*|$$
$$\geqslant L\,|x_k - x_*|$$

implying $x_{k+m} \notin I \quad \exists m \in \mathbb{Z}^+$.

This shows that no matter how large k is
$$|x_{\tilde{k}} - x_*| \geqslant \delta \quad \exists \tilde{k} > k,$$
so (x_k) does not converge to x_*. $\square$

Ex

$$h(x) = \frac{x^2+2}{2x} \qquad h'(\sqrt{2}) = 0$$

$\sqrt{2}$ is a stable fixed-point.

$$g(x) = x^2 - 2 + x \qquad g'(\sqrt{2}) = 1 + 2\sqrt{2}$$

$\sqrt{2}$ is an unstable fixed-point

($|g'(-\sqrt{2})| = |1 - 2\sqrt{2}| > 1$

$-\sqrt{2}$ is also unstable)

Newton's method

Consider

$$g(x) = x - \lambda(x) f(x)$$

where $\lambda(x)$ is continuous in a neighborhood of x_*.

(x_k), $\quad x_{k+1} = x_k - \lambda(x_k) f(x_k)$

Suppose $\lim_{k \to \infty} x_k = x_*$, then $g(x_*) = x_*$, and

$$x_* = g(x_*) = x_* - \lambda(x_*) f(x_*)$$

$$\Longrightarrow f(x_*) = 0 \quad \text{provided } \lambda(x_*) \neq 0.$$

Choose $\lambda(x)$ so as to minimize $|g'(x_*)|$,

$$|g'(x_*)| = |1 - \lambda(x_*) f'(x_*)|$$

For instance, choose

$$\lambda(x) = 1/f'(x)$$

leading to

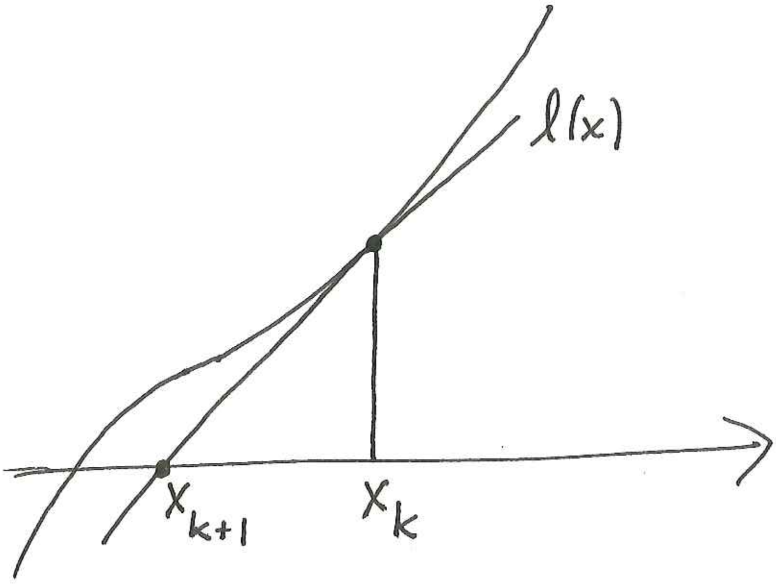
$$g(x) = x - f(x)/f'(x)$$

Newton's method generates the sequence (x_k)

$$x_{k+1} = x_k - f(x_k)/f'(x_k)$$

Geometrically,

$$\ell(x_{k+1}) = 0, \quad \text{where } \ell(x) = f(x_k) + f'(x_k)(x - x_k)$$



Ex

$f(x) = x^2 - 2$

Newton sequence $\quad x_{k+1} = x_k - \dfrac{x_k^2 - 2}{2x_k}$

$$= \frac{x_k^2 + 2}{2x_k}.$$

Defn (Higher Order Convergence)

Suppose that $\lim_{k\to\infty} x_k = x_*$, and there exists (ε_k) such that $|x_k - x_*| \leq \varepsilon_k$

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^q} = M$$

for some $q > 1$ and $M > 0$.

(i) (x_k) converges to x_* with at least order q.

(ii) If, additionally $\varepsilon_k = |x_k - x_*|$, x_k converges to x_* with order q, in particular quadratically if $q = 2$.

Ex

$\{10^{-2^k}\}$ converges to 0 quadratically.

$$\lim_{k\to\infty} \frac{10^{-2^{k+1}}}{(10^{-2^k})^2} = \lim_{k\to\infty} \frac{10^{-2^{k+1}}}{10^{-2^{k+1}}} = 1$$

$\{10^{-p^k}\}$, $p>1$ converges to 0 with order p.

THM

Let $x_* \in \mathbb{R}$ be such that $f(x_*)=0$ and $f''(x)$ is continuous on $[x_*-\delta, x_*+\delta]$. Furthermore, suppose $f'(x_*) \neq 0$, $f''(x_*) \neq 0$, and

$$\left| \frac{f''(x)}{f'(y)} \right| \leq A \quad \forall x,y \in [x_*-\delta, x_*+\delta].$$

The Newton sequence (x_k) converges to x_* quadratically for all $x_0 \in [x_*-h, x_*+h]$, $h := \min\{\delta, 1/A\}$.

Proof

Suppose $x_k \in [x_*-h, x_*+h]$. By Taylor's thm

$$0 = f(x_*) = f(x_k) + f'(x_k)(x_* - x_k) + \frac{f''(\xi_k)}{2}(x_k - x_*)^2$$

$$\Longrightarrow \quad 0 = \underbrace{\frac{f(x_k)}{f'(x_k)}}_{x_k - x_{k+1}} + (x_* - x_k) + \frac{f''(\xi_k)(x_k-x_*)^2}{2f'(x_k)}$$

$$\Longrightarrow \quad (x_{k+1} - x_*) = \frac{f''(\xi_k)(x_k-x_*)^2}{2f'(x_k)}$$

$$\Longrightarrow \quad |x_{k+1} - x_*| \leq \frac{1}{2A} \quad \Longrightarrow \quad x_{k+1} \in [x_*-h, x_*+h],$$

where ξ_k is in the open interval with end-points x_k, x_*.
By induction $x_k \in [x_*-h, x_*+h]$ $\forall k$.

Moreover,

$$|x_{k+1} - x_*| \leqslant \frac{1}{2} \underbrace{\left|\frac{f''(\xi_k)}{f'(x_k)}\right|}_{\leqslant A} \underbrace{|x_k - x_*|}_{\leqslant 1/A} |x_k - x_*|$$

$$\leqslant 1/2 \, |x_k - x_*| .$$

By induction $|x_k - x_*| \leqslant 2^{-k} |x_0 - x_*|$ implying $\lim_{k\to\infty} x_k = x_*$. Finally,

$$\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|^2} = \lim_{k\to\infty} \left|\frac{f''(\xi_k)}{2f'(x_k)}\right|$$

$$= \frac{1}{2} \left| f''(x_*) / f'(x_*) \right| .$$

proving quadratic convergence. □

Secant Method (a quasi-Newton method)

Approximates $f'(x_k)$ in Newton's method with

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

Secant method generates a sequence (x_k) s.t.

$$x_{k+1} = x_k - \frac{f(x_k)\,(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

(given x_0, x_1)

* Does not fit into $x_{k+1} = g(x_k)$
* Instead fits into $x_{k+1} = g(x_k, x_{k-1}, \ldots, x_{k-\ell})$ with $\ell = 1$.

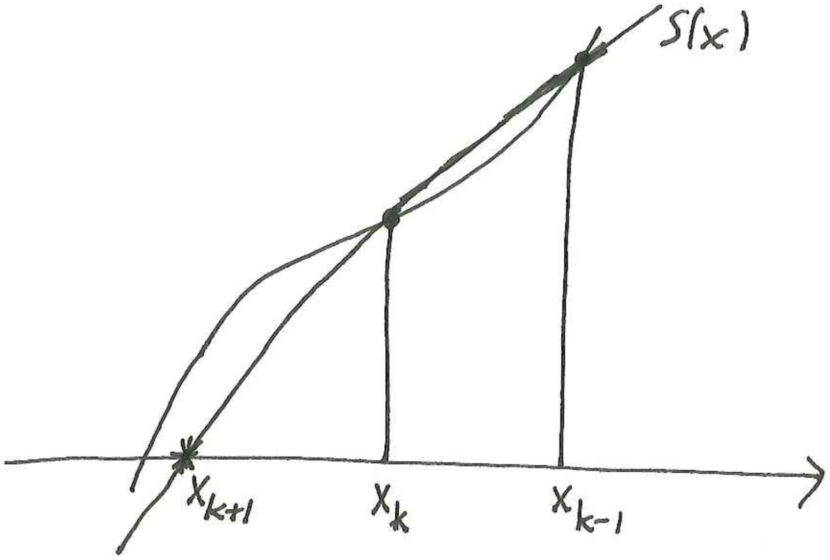
Ex

$f(x) = x^2 - 2$

sequence by secant method
$$x_{k+1} = x_k - \frac{(x_k^2-2)(x_k - x_{k-1})}{x_k^2 - x_{k-1}^2}$$
$$= x_k - \frac{x_k^2 - 2}{x_k + x_{k-1}}$$

Geometrically

$s(x_{k+1}) = 0$ where $s(x) = f(x_k) + \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}(x - x_{k-1})$



THM

Let $x_* \in \mathbb{R}$ be such that $f(x_*) = 0$ and $f'(x)$ is continuous in a neighborhood of x_*. Furthermore, suppose $f'(x_*) \neq 0$.

(i) The sequence (x_k) by secant method satisfies $\lim_{k\to\infty} x_k = x_*$ for all x_0, x_1 sufficiently close to x_*.

(ii) The order of this convergence is superlinear, provided $f'(x)$ is Lipschitz continuous in a neighborhood of x_*.

Proof (WLOG assume $f'(x_*) > 0$.)

(i) Letting $\alpha := f'(x_*) > 0$, $\exists\, \delta > 0$ such that

$$0 < \frac{3\alpha}{4} \leq f'(x) \leq \frac{5\alpha}{4} \qquad \forall x \in [x_* - \delta, x_* + \delta].$$

By the secant update rule, assuming $x_k, x_{k-1} \in [x_*-\delta, x_*+\delta]$,

$$(x_* - x_{k+1}) = (x_* - x_k) + \frac{f(x_k)\,(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

$$= (x_* - x_k) + \frac{f'(n_k)\,(x_k - x_*)}{f'(\epsilon_k)}$$

for some $n_k, \epsilon_k \in (x_* - \delta, x_* + \delta)$. Hence,

$$|x_* - x_{k+1}| = |x_* - x_k| \left| 1 - \frac{f'(n_k)}{f'(\epsilon_k)} \right|$$

$$\leq 2/3\, |x_* - x_k|$$

implying $x_k \in [x_* - \delta, x_* + \delta]$. Indeed,

$$|x_* - x_k| \leq (2/3)^{k-1} |x_* - x_1| \implies \lim_{k \to \infty} x_k = x_*.$$

(ii) By FTC

$$0 = f(x_*) = f(x_k) + \int_{x_k}^{x_*} f'(x)\, dx$$

$$= f(x_k) + \int_0^1 f'(x_k + t(x_* - x_k))\,(x_* - x_k)\, dt$$

Letting $g(x_k, x_{k-1}) = \{f(x_k) - f(x_{k-1})\} / \{x_k - x_{k-1}\}$,

$$0 = \frac{f(x_k)}{\substack{\cancel{g(x_k, x_{k-1})} \\ f'(x_k)}} + \frac{1}{\substack{\cancel{g(x_k, x_{k-1})} \\ f'(x_k)}} \int_0^1 f'(x_k + t(x_* - x_k))\,(x_* - x_k)\, dt$$

$$= \underbrace{\frac{f(x_k)}{g(x_k, x_{k-1})}}_{x_k - x_{k+1}} + f(x_k)\left\{\frac{1}{f'(x_k)} - \frac{1}{g(x_k, x_{k-1})}\right\} + (x_* - x_k)$$

$$+ \frac{1}{f'(x_k)} \int_0^1 \left\{f'(x_k + t(x_* - x_k)) - f'(x_k)\right\}(x_* - x_k)\,dt$$

$$= (x_* - x_{k+1}) + f(x_k)\left\{\frac{1}{f'(x_k)} - \frac{1}{g(x_k, x_{k-1})}\right\}$$

$$+ \frac{1}{f'(x_k)} \int_0^1 \left\{f'(x_k + t(x_* - x_k)) - f'(x_k)\right\}(x_* - x_k)\,dt$$

Using $|a+b| \leq |a| + |b|$, we have

$$|x_* - x_{k+1}| \leq |f'(\varepsilon_k)|\,|x_* - x_k|\,\frac{|g(x_k, x_{k-1}) - f'(x_k)|}{|f'(x_k)|\,|g(x_k, x_{k-1})|}$$

$$+ \frac{1}{|f'(x_k)|} \int_0^1 t\gamma\,|x_* - x_k|^2\,dt$$

where ε_k is in the open interval with end-points x_*, x_k, and γ is the Lipschitz constant for $f'(x)$ (i.e. $|f'(x) - f'(y)| \leq \gamma|x-y|$ $\forall x, y$ close to x_*). Hence,

$$\lim_{k \to \infty} \frac{|x_* - x_{k+1}|}{|x_* - x_k|} = \lim_{k \to \infty} |f'(\varepsilon_k)|\,\frac{|g(x_k, x_{k-1}) - f'(x_k)|}{|f'(x_k)|\,|g(x_k, x_{k-1})|}$$

$$+ \lim_{k \to \infty} \frac{1}{|f'(x_k)|}\,\frac{\gamma\,|x_* - x_k|}{2} = 0$$

as desired. $\square$