CH8 - P2

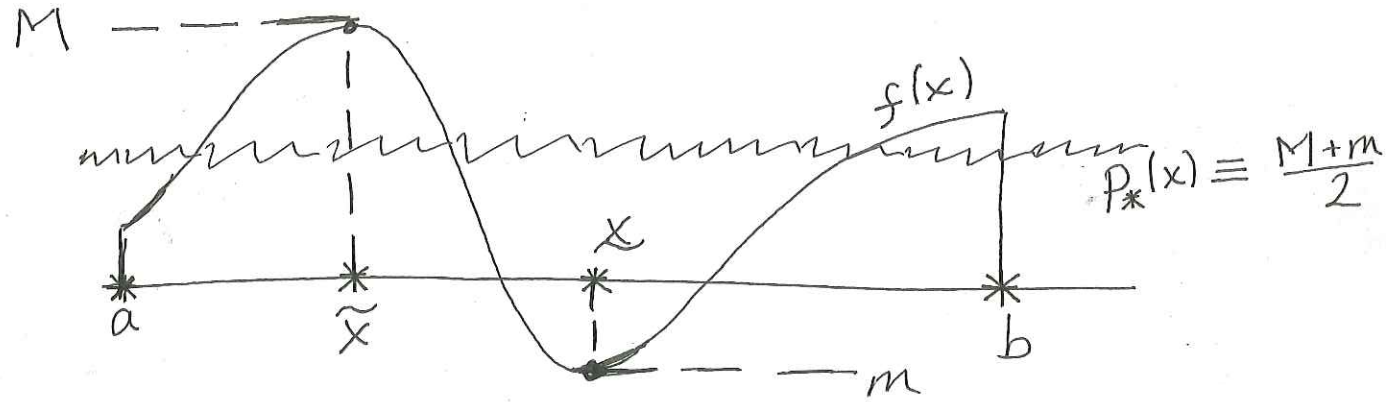
Best approximation in the ∞-norm

Find $p_* \in \mathcal{P}_n$ such that

$$\| f - p_* \|_\infty = \min_{p_n \in \mathcal{P}_n} \| f - p_n \|_\infty$$

(where $f \in C[a,b]$)

Case $n = 0$



Let M, m be given by

$$M := \max_{x \in [a,b]} f(x), \qquad m := \min_{x \in [a,b]} f(x)$$

and $\tilde{x}, \underset{\sim}{x} \in [a,b]$ be such that

$$f(\tilde{x}) = M, \quad f(\underset{\sim}{x}) = m.$$

$p_0(x) \equiv c$

if $c < \frac{M+m}{2}$

$$|f(\tilde{x}) - p_0(\tilde{x})| > \frac{M-m}{2} \quad \left(\Longrightarrow \|f - p_0\|_\infty > \frac{M-m}{2} \right.$$

if $\quad c > \frac{M+m}{2}$

$$|f(\underset{\sim}{x}) - p_0(\underset{\sim}{x})| > \frac{M-m}{2} \quad \left(\Longrightarrow \|f - p_0\| > \frac{M-m}{2}\right.$$

for $\quad c = \frac{M+m}{2}$

$$\|f - p_0\|_\infty = \frac{M-m}{2}$$

Hence, $p_* \equiv \frac{M+m}{2}$ satisfies

$$\|f - p_*\|_\infty = \min_{p_0 \in P_0} \|f - p_0\|_\infty .$$

Observe

(i) $f(\tilde{x}) - p_*(\tilde{x}) = \|f - p_*\|_\infty$

(ii) $f(\underset{\sim}{x}) - p_*(\underset{\sim}{x}) = -\|f - p_*\|_\infty$

<u>THM</u> (De la Vallée Poussin's Theorem)

Let $p \in P_n$ and $x_0 < x_1 < \cdots < x_n < x_{n+1}$ be all in $[a, b]$ such that $f(x_j) - p(x_j)$ and $f(x_{j+1}) - p(x_{j+1})$ have opposite signs for $j = 0, 1, \cdots, n$. Then

$$\min_{p_n \in P_n} \|f - p_n\|_\infty \geq \min_{j=0,\cdots,n+1} |f(x_j) - p(x_j)|$$

Proof

Letting $\mu := \min\limits_{j=0,\dots,n+1} |f(x_j) - p(x_j)|$, if $\mu = 0$, there is nothing to prove. Hence, assume $\mu > 0$.

For the sake of contradiction, suppose

$$\|f - p_*\|_\infty = \min_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty < \mu$$

where $p_* \in \mathcal{P}_n$. But then

$$(*) \quad |f(x_j) - p_*(x_j)| < |f(x_j) - p(x_j)| \quad j = 0,\dots,n+1.$$

Furthermore,

$$p(x_j) - p_*(x_j) = \{p(x_j) - f(x_j)\} - \{p_*(x_j) - f(x_j)\}$$

and $(*)$ implies

$$p(x_j) - p_*(x_j), \quad p(x_j) - f(x_j)$$

have the same sign for $j = 0, \dots, n+1$.

Consequently, $p(x_j) - p_*(x_j)$ and $p(x_{j+1}) - p_*(x_{j+1})$ have opposite signs implying

$$p(\eta_j) - p_*(\eta_j) = 0 \quad \exists \eta_j \in (x_j, x_{j+1})$$

for $j = 0, 1, \dots, n$. But, as $h \equiv p - p_* \in \mathcal{P}_n$, this implies $p \equiv p_*$ which contradicts $(*)$. $\square$

Defn (Minimax Polynomial)

A polynomial $p_* \in \mathcal{P}_n$ such that

$$\|f - p\|_\infty = \min_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty$$

$$= \min_{p_n \in \mathcal{P}_n} \max_{x \in [a,b]} |f(x) - p_n(x)|$$

is called a minimax polynomial for f of degree n.

THM (Theorem of Alternations)

Let $f \in C[a,b]$, $p \in \mathcal{P}_n$. The following are equivalent:

(1) p is a minimax polynomial for f of degree n.

(2) There exist $x_0 < x_1 < \dots < x_{n+1}$ contained in $[a,b]$ such that

(i) $f(x_j) - p(x_j) = -\{f(x_{j+1}) - p(x_{j+1})\}$ for $j = 0, 1, \dots, n$,

(ii) $|f(x_j) - p(x_j)| = \|f - p\|_\infty$ for $j = 0, 1, \dots, n+1$

Proof of (2) $\Longrightarrow$ (1)

By de la Vallée Poussin's thm

$$\min_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty \geqslant \|f - p\|_\infty,$$

additionally as $p \in \mathcal{P}_n$, we have

$$\min_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty \leqslant \|f - p\|_\infty$$

proving (1).

Ex

Let $f \in C[a,b]$ be such that $f'(x)$ exists, is continuous and monotonically increasing on $[a,b]$.

Find minimax polynomial $p_* \in \mathcal{P}_1$ of degree one for f.

$|f(x) - p_*(x)|$ can be maximized only at a, b and the unique $c \in (a,b)$ such that

$$f'(c) - p_*'(c) = 0$$

(uniqueness of c due to monotonicity of f')

As x_0, x_1, x_2 in the thm of alternations have to be maximizers of $|f(x) - p_*(x)|$,

$$x_0 = a, \quad x_1 = c, \quad x_2 = b.$$

Letting $p_*(x) = a_1 x + a_0$ for $a_1, a_0 \in \mathbb{R}$ to be determined, and $L := \|f - p_*\|_\infty$, we have

$$\begin{aligned}
&(1) \quad f(a) - \{a_1 a + a_0\} = A \\
&(2) \quad f(c) - \{a_1 c + a_0\} = -A \\
&(3) \quad f(b) - \{a_1 b + a_0\} = A,
\end{aligned}$$

$\exists A$ s.t. $|A| = L$

additionally

$$(4) \quad f'(c) = p_*'(c) = a_1.$$

Subtracting (1) from (3) yields

$$\text{(i)} \quad a_1 = \{f(b) - f(a)\} / \{b-a\}.$$

Moreover, by the MVT and (4), c is uniquely defined by

$$\text{(ii)} \quad f'(c) = \{f(b) - f(a)\} / \{b-a\}.$$

Adding (1) and (2), we have

$$\text{(iii)} \quad a_0 = \frac{\{f(a) + f(c)\} - a_1 \{a+c\}}{2}$$

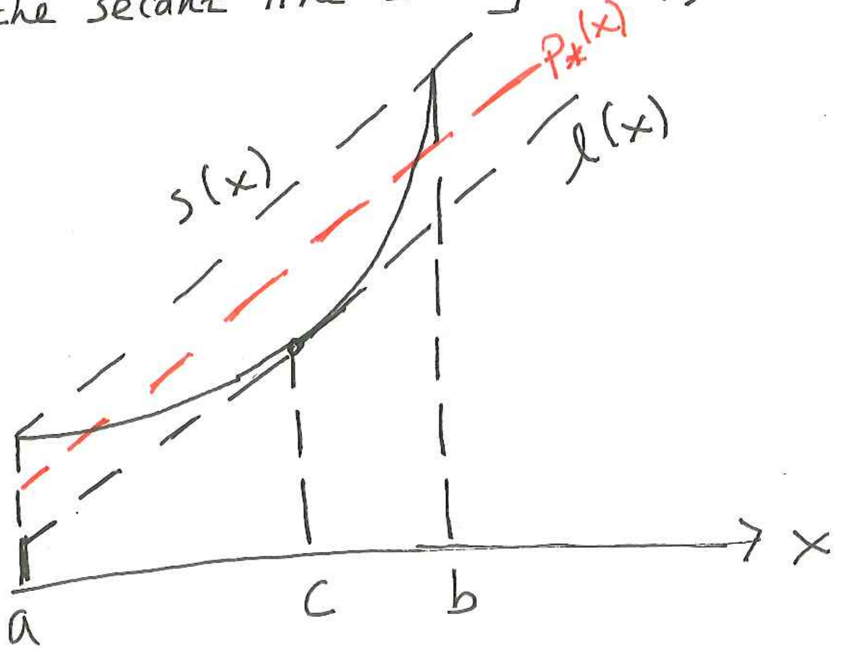
Hence,

$$P_*(x) = f'(c)x + \frac{\{f(a)+f(c)\} - f'(c)\{a+c\}}{2}$$

$$= f'(c)(x-a) + \frac{\{f(a)+f(c)\} + f'(c)\{a-c\}}{2}$$

$$= f'(c)(x-a) + \frac{l(a) + s(a)}{2}$$

where $l(x)$ is the tangent line for $f(x)$ at c, $s(x)$ is the secant line through $(a, f(a))$ and $(b, f(b))$.



THM (Uniqueness)

There exists a unique polynomial $p_* \in \mathcal{P}_n$ such that

$$\|f - p_*\|_\infty = \min_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty .$$

Proof

Suppose $p_*, q_* \in \mathcal{P}_n$ are such that

$$E_n(f) := \|f - p_*\|_\infty = \|f - q_*\|_\infty = \min_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty .$$

But then, by triangle inequality and homogeneity

$$\left\|f - \frac{p_* + q_*}{2}\right\|_\infty = \left\|\left(\frac{f - p_*}{2}\right) + \left(\frac{f - q_*}{2}\right)\right\|_\infty$$

$$\leq \frac{1}{2}\|f - p_*\|_\infty + \frac{1}{2}\|f - q_*\|_\infty$$

$$= E_n(f)$$

$$\Longrightarrow \left\|f - \left(\frac{p_* + q_*}{2}\right)\right\|_\infty = E_n(f).$$

By the thm of alternations, there exist $x_0, \dots, x_{n+1}$ on $[a, b]$ such that

$$\left|f(x_j) - \tfrac{1}{2}(p_* + q_*)(x_j)\right| = E_n(f) \quad j = 0, 1, \dots, n+1$$

$$\Longrightarrow (+) \left|\{f(x_j) - p_*(x_j)\} + \{f(x_j) - q_*(x_j)\}\right| = 2E_n(f) \; j = 0, \dots, n+1$$

But as $\|f - p_*\|_\infty = \|f - q_*\|_\infty = E_n(f)$, we have

$$(++) \; |f(x_j) - p_*(x_j)| \leq E_n(f), \quad |f(x_j) - q_*(x_j)| \leq E_n(f)$$

for $j = 0, \dots, n+1$. Combining $(+)$ and $(++)$, we deduce

$$f(x_j) - p_*(x_j) = f(x_j) - q_*(x_j) = E_n(f), \text{ OR}$$

$$f(x_j) - p_*(x_j) = f(x_j) - q_*(x_j) = -E_n(f).$$

In any case, $h_* = p_* - q_* \in \mathcal{P}_n$ has $(n+2)$ roots, namely $x_0, \dots, x_{n+1}$. But this means $h_* \equiv 0$, equivalently $p_* \equiv q_*$. □

Theorem of Alternations (page ④)

<u>Justification for (1) $\Longrightarrow$ (2)</u>

Suppose $n = 2$ (use contrapositive $\sim(2) \Longrightarrow \sim(1)$)

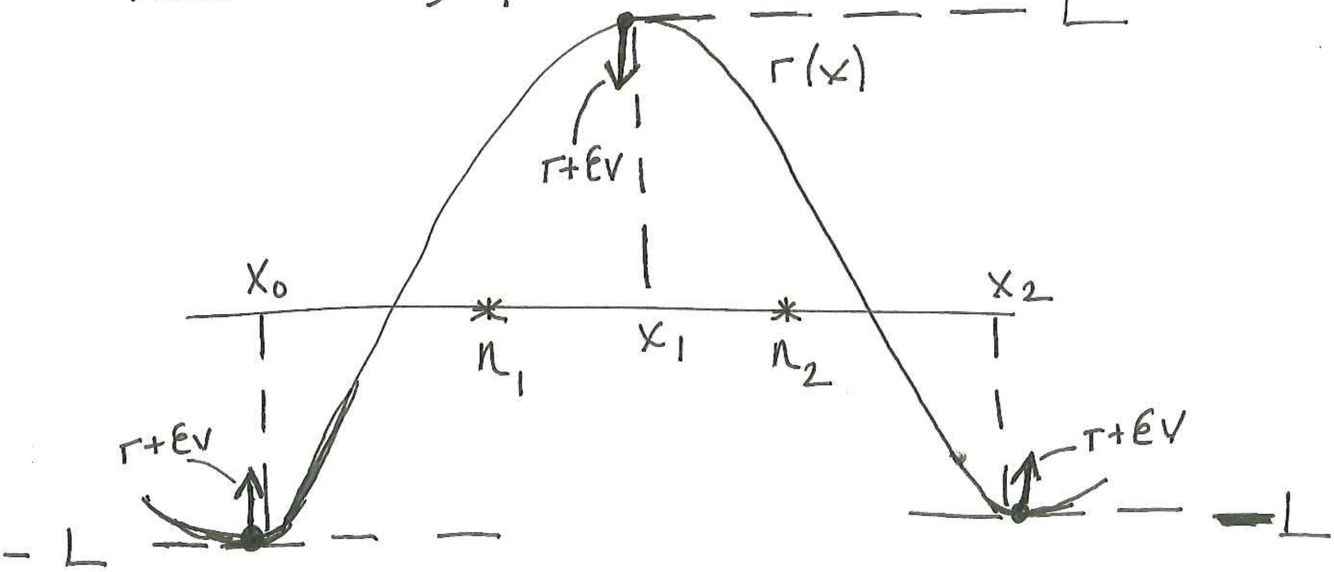
There are only 3 points x_0, x_1, x_2 on $[a,b]$ such that

(i) $\{f(x_0) - p(x_0)\} = -\|f-p\|_\infty$

(ii) $\{f(x_1) - p(x_1)\} = \|f-p\|_\infty =: L$

(iii) $\{f(x_2) - p(x_2)\} = -\|f-p\|_\infty$

Let $r := f - p$



Let $n_1 \in (x_0, x_1)$ and $n_2 \in (x_1, x_2)$.

Define $v(x) := (n_1 - x)(n_2 - x) \in P_2$

$p_* := p - \varepsilon v$ (for $\varepsilon > 0$ sufficiently small)

$r_* := f - p_* = r + \varepsilon v$

$\|r_*\|_\infty < L$ for all $\varepsilon > 0$ small enough

$\Longrightarrow$ p is not the minimax polynomial.

(This argument generalizes to any n as long as # alternation points $\leq n+1$.)

Chebyshev Polynomials

$$T_n(x) = \cos(n \arccos(x)) \qquad x \in [-1, 1]$$

$n = 0, 1, 2, 3, \ldots$

$$T_0(x) \equiv 1$$

$$T_1(x) = \cos(\arccos(x)) = x$$

$T_n(x)$ for $n \geq 2$ are also indeed polynomial

Trigonometric identities

$$\begin{cases} \cos(a+b) = \cos a \cos b - \sin a \sin b \\ \cos(a-b) = \cos a \cos b + \sin a \sin b \end{cases}$$

$$\Longrightarrow$$

$$(*) \cos(a+b) + \cos(a-b) = 2 \cos a \cos b$$

Letting $\theta := \arccos x$ (that is $x = \cos\theta$)

$$T_{n+1}(x) + T_{n-1}(x)$$

$$= \cos((n+1)\theta) + \cos((n-1)\theta)$$

$$\overset{\text{From } (*)}{=} 2 \cos(n\theta) \cos\theta = 2x T_n(x)$$

3-term Recursion

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

$n = 1, 2, 3, \ldots$

$$T_2(x) = 2x\,T_1(x) - T_0(x)$$
$$= 2x^2 - 1$$

$$T_3(x) = 2x\,T_2(x) - T_1(x)$$
$$= 2x(2x^2-1) - x$$
$$= 4x^3 - 3x$$

Lemma (Properties of Chebyshev Polynomials)

(1) $T_n(x)$ is even if n is even,
$T_n(x)$ is odd if n is odd.

(2) The leading term of $T_n(x)$ is $2^{n-1}x^n$
for $n = 1, 2, 3, \ldots$

(3) $|T_n(x)| \leq 1 \quad \forall x \in [-1, 1]$

(4) $|T_n(x_j)| = 1$ for $x_j = \cos\left(\frac{j\pi}{n}\right)$ $j = 0, \ldots, n$,
furthermore $T_n(x_j) = -T_n(x_{j+1})$ $j = 0, \ldots, n-1$.

(5) $T_n(x_j) = 0$ for $x_j = \cos\left(\frac{(2j+1)\pi}{2n}\right)$ $j = 0, \ldots, n-1$.

Consider again minimizing

$$\|\pi_{n+1}\|_\infty \qquad \pi_{n+1}(x) := (x-x_0)\cdots(x-x_n)$$

over $x_0, x_1, \ldots, x_n$ (in $[-1,1]$).

Equivalently find $\pi_{n+1} \in \mathcal{P}'_{n+1}$ with

$$\|\pi_{n+1}\|_\infty$$

is as small as possible.

$$\left(\mathcal{P}'_{n+1} := \left\{ \overbrace{c_0 + c_1 x + \cdots + c_n x^n}^{-p_n(x) \in \mathcal{P}_n} + x^{n+1} \;\middle|\; c_0, \ldots, c_n \in \mathbb{R} \right\}\right)$$

set of monic polynomials of degree $(n+1)$

Equivalently find $p_* \in \mathcal{P}_n$ s.t.

$$\|x^{n+1} - p_*\|_\infty = \min_{p_n \in \mathcal{P}_n} \|x^{n+1} - p_n\|_\infty$$

<u>THM</u>

The minimax polynomial of degree n for $f(x) = x^{n+1}$ on $[-1,1]$ is

$$(+) \quad p_*(x) = x^{n+1} - 2^{-n} T_{n+1}(x).$$

<u>Proof</u>

First observe p_* as in $(+)$ belongs to $\mathcal{P}_n$, as the leading ~~coefficient~~ term for T_{n+1} is $2^n x^{n+1}$!

Furthermore, for p_* as in $(+)$

$$\|x^{n+1} - p_*\|_\infty = \|2^{-n} T_{n+1}(x)\|_\infty = 2^{-n},$$

and $\|2^{-n} T_{n+1}(x_j)\|_\infty = 2^{-n}$ for $x_j = \cos\left(\frac{j\pi}{n+1}\right)$ $j = 0, \ldots, n+1$

as well as $2^{-n} T_{n+1}(x_j) = -\{2^{-n} T_{n+1}(x_{j+1})\}$ for $j = 0, \ldots, n$.

By the theorem of alternations, p_* is the minimax polynomial of degree n for $f(x) = x^{n+1}$. $\square$

Observe also

$$\min_{\pi_{n+1} \in \mathcal{P}'_{n+1}} \|\pi_{n+1}\|_\infty = \min_{p_n \in \mathcal{P}_n} \|x^{n+1} - p_n\|_\infty$$

$$= \|x^{n+1} - p_*\|_\infty \quad \left(\begin{array}{c}\text{where}\\ p_* \text{ as}\\ \text{in } (+)\end{array}\right)$$

$$= \|\underbrace{2^{-n} T_{n+1}}_{\in \mathcal{P}'_{n+1}}\|_\infty .$$

Optimal interpolation points (w.r.t. ∞-norm) are

* roots of $T_{n+1}(x)$
* that is $x_j = \cos\left(\frac{(2j+1)\pi}{2n+2}\right)$ $j = 0, \dots, n$.

Ex

$$\pi_2(x) = (x - x_0)(x - x_1)(x - x_2)$$

Choosing

x_0, x_1, x_2

roots of

$$T_3(x) = 4x^3 - 3x$$

that is $x_0 = -\sqrt{3}/2$, $x_1 = 0$, $x_2 = \sqrt{3}/2$

minimizes $\|\pi_2(x)\|_\infty$