Simultaneous Nonlinear Equations

Find $x \in \mathbb{R}^n$ such that

$$\begin{aligned} f_1(x) &= 0 \\ f_2(x) &= 0 \\ &\vdots \\ f_n(x) &= 0 \end{aligned}$$

$f_1, f_2, \ldots, f_n : \mathbb{R}^n \to \mathbb{R}$.

Equivalently, find x s.t.

$$f(x) = 0$$

$f : \mathbb{R}^n \to \mathbb{R}^n$, $f_j(x)$ − jth component of $f(x)$

Ex

① $$2x_1^2 - 2x_1x_2 + x_2^2 - 13/200 = 0$$
$$x_1^2 + x_1x_2 + x_2^2 - 61/400 = 0$$

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) = \begin{bmatrix} 2x_1^2 - 2x_1x_2 + x_2^2 - 13/200 \\ x_1^2 + x_1x_2 + x_2^2 - 61/400 \end{bmatrix}$$

(Check $x_* = (1/4, 1/5)$, $f(x_*) = 0$)

② Eigenvalue problem

$$\underbrace{A}_{n\times n} v = \lambda v \quad \exists v \neq 0$$

$$f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad f(v,\lambda) = \begin{bmatrix} Av - \lambda v \\ v^T v - 1 \end{bmatrix}$$

Find (v,λ) such that $f(v,\lambda) = 0$.

Linear systems is a special case,

$$Ax = b$$

$$\Longleftrightarrow$$

$$f(x) = Ax - b = 0.$$

Fixed point problem

$x \in \mathbb{R}^n$ is a fixed point of $g : \mathbb{R}^n \to \mathbb{R}^n$

if $g(x) = x$.

$$f(x) = 0 \quad \Longleftrightarrow \quad x = \underbrace{f(x) + x}_{g(x)}$$

$$\Longleftrightarrow \quad x = -\lambda f(x) + x \quad \forall \lambda \neq 0$$

Simultaneous Iteration (or Fixed Point Iteration)

$\{x^{(k)}\}$ in $\mathbb{R}^n$ defined by

$$x^{(k+1)} = g(x^{(k)})$$

for a given $x^{(0)} \in \mathbb{R}^n$.

Notation: $x_j^{(k)}$ – jth component of $x^{(k)}$.

For Ex 1 on page 1

define $g : \mathbb{R}^2 \to \mathbb{R}^2$

$$g(x_1, x_2) = (x_1, x_2)^T - f(x_1, x_2)$$

Associated simultaneous iteration

$$x_1^{(k+1)} = x_1^{(k)} - 2\left[x_1^{(k)}\right]^2 + 2x_1^{(k)}x_2^{(k)} - \left[x_2^{(k)}\right]^2 + 13/200$$

$$x_2^{(k+1)} = x_2^{(k)} - \left[x_1^{(k)}\right]^2 - \left[x_1^{(k)}\right]\left[x_2^{(k)}\right] - \left[x_2^{(k)}\right]^2 + 61/400$$

For linear system $Ax = b$

(and $f(x) = Ax - b$)

$$Ax = b \iff Dx = (-L-U)x + b$$

$$\iff x = D^{-1}(-L-U)x + D^{-1}b$$

where $A = L + U + D$, $L \in \mathbb{R}^{n \times n}$ is the lower triangular part of A, $U \in \mathbb{R}^{n \times n}$ is the upper triangular part, $D \in \mathbb{R}^{n \times n}$ is the diagonal part (assuming $d_{ii} \neq 0$ $i = 1, \ldots, n$)

Associated simultaneous iteration

$$x^{(k+1)} = D^{-1}(-L-U)x^{(k)} + D^{-1}b$$

(known as Jacobi iteration)

Questions as before

(i) Does g have a fixed point?

(ii) If yes, is it unique?

(iii) Is the simultaneous iteration $\{x^{(k)}\}$ in $\mathbb{R}^n$ convergent?

Defn (Lipschitz Continuity)

A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is called Lipschitz continuous on a subset D (of $\mathbb{R}^n$) in the ∞-norm if

$$(+) \quad \| g(x) - g(y) \|_\infty \leqslant \gamma \| x - y \|_\infty \quad \forall x, y \in D$$

for some $\gamma > 0$.

The constant γ is called the Lipschitz constant for g in the ∞-norm.

Defn (Contraction)

A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is called a contraction on a subset D of $\mathbb{R}^n$ in the ∞-norm if $(+)$ holds for some $\gamma \in (0,1)$.

Ex

$$\underline{g}(x) = \begin{bmatrix} 2/5 & 3/10 \\ -7/10 & 7/20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\| \underline{g}(x) - \underline{g}(\underline{y}) \|_\infty = \left\| \begin{bmatrix} 2/5 & 3/10 \\ -7/10 & 7/20 \end{bmatrix} (x - \underline{y}) \right\|_\infty$$

$$\leq \underbrace{\left\| \begin{bmatrix} 2/5 & 3/10 \\ -7/10 & 7/20 \end{bmatrix} \right\|_\infty}_{21/20} \| x - \underline{y} \|$$

is Lipschitz continuous on $\mathbb{R}^n$ in the ∞-norm with Lipschitz constant $\gamma = 21/20$, but not a contraction.

Remark

If $\underline{g} : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous on ~~$\mathbb{R}^n$~~ D (in the ∞-norm), it is also continuous on D.

i.e.,

Suppose
$$\| \underline{g}(x) - \underline{g}(\underline{y}) \|_\infty \leq \gamma \| x - \underline{y} \|_\infty \quad \forall x, \underline{y} \in D.$$

Then for each ε, letting $\delta := \varepsilon / \gamma$ we have for every $\underline{y} \in D$
$$\| x - \underline{y} \|_\infty \leq \delta \implies \| \underline{g}(x) - \underline{g}(\underline{y}) \|_\infty \leq \varepsilon .$$

Hence, for each ε and $\delta = \varepsilon / \gamma$
$$\| \underline{g}(x) - \underline{g}(\underline{y}) \|_\infty \leq \varepsilon \quad \forall x \in D \cap \overline{B}(\underline{y}, \delta)$$
implying continuity at $\underline{y}$. (Here, $\overline{B}(\underline{y}, \delta) := \{ x \in \mathbb{R}^n \mid \| x - \underline{y} \|_\infty \leq \delta \}$)

THM (Contraction Mapping Theorem)

Let $g : D \to \mathbb{R}^n$ be such that $g(D) \subseteq D$, where D is a closed subset of $\mathbb{R}^n$. Additionally, suppose g is a contraction on D in the ∞-norm.

Then

(i) g has a unique fixed point in D,

(ii) $\{x^{(k)}\}$, $x^{(k+1)} = g(x^{(k)})$ converges to this unique fixed point for all $x^{(0)} \in D$.

Ex

Jacobi iteration (see page ③)

$$\{x^{(k)}\}, \quad x^{(k+1)} = D^{-1}(-L-U)\,x^{(k)} + D^{-1}b$$

Suppose A is diagonally dominant, that is

$$|a_{jj}| > \sum_{\substack{k=1 \\ k\neq j}}^{n} |a_{jk}|$$

for $j = 1, \ldots, n$.

Then $\|D^{-1}(-L-U)\|_\infty < 1$. Consider

$$g(x) = D^{-1}(-L-U)\,x + D^{-1}b.$$

Since

$$\|g(x) - g(y)\|_\infty = \|D^{-1}(-L-U)\,(x-y)\|_\infty \leq \|D^{-1}(-L-U)\|_\infty \, \|x-y\|_\infty$$

for all $x, y \in \mathbb{R}^n$;

by Contraction Mapping Thm,

$$\lim_{k\to\infty} x^{(k)} = x_*$$

where x_* is s.t.

$$x_* = D^{-1}(-L-U)\,x_* + D^{-1}b$$

$$\Longleftrightarrow$$

$$Ax_* = b.$$

Proof

(i) Observe that if $x^{(k)} \in D$, so does $x^{(k+1)} = g(x^{(k)})$ as $g(D) \subseteq D$. Hence,

$$x^{(0)} \in D \implies x^{(k)} \in D \quad \forall k.$$

Now we shall show that $\{x^{(k)}\}$ is convergent for all $x^{(0)} \in D$.

We have

$$\begin{aligned} \|x^{(k+1)} - x^{(k)}\|_\infty &= \|g(x^{(k)}) - g(x^{(k-1)})\|_\infty \\ &\leqslant \gamma \|x^{(k)} - x^{(k-1)}\|_\infty \\ &\leqslant \gamma^k \|x^{(1)} - x^{(0)}\|_\infty \end{aligned}$$

where $\gamma < 1$ (as g is a contraction) is the Lipschitz constant on D. By triangle inequality for $m > k$

$$\begin{aligned} \|x^{(m)} - x^{(k)}\|_\infty &\leqslant \|x^{(m)} - x^{(m-1)}\|_\infty + \|x^{(m-1)} - x^{(m-2)}\|_\infty \\ &\quad + \dots + \|x^{(k+1)} - x^{(k)}\|_\infty \\ &\leqslant (\gamma^{m-1} + \gamma^{m-2} + \dots + \gamma^k)\, \|x^{(1)} - x^{(0)}\|_\infty \\ &= \frac{\gamma^k(1-\gamma^{m-k})}{(1-\gamma)} \|x^{(1)} - x^{(0)}\|_\infty. \end{aligned}$$

$$\leqslant \gamma^k/_{(1-\gamma)} \|x^{(1)} - x^{(0)}\|_\infty$$

Since $\gamma^{(k)}/_{(1-\gamma)} \to 0$ as $k \to \infty$, the inequality above implies that the sequence $\{x^{(k)}\}$ is Cauchy. In particular, for any ϵ choose the integer K so that $\gamma^K/_{(1-\gamma)} \|x^{(1)}-x^{(0)}\|_\infty \leqslant \epsilon$, implying

$$\|x^{(m)} - x^{(k)}\|_\infty \leqslant \gamma^k/(1-\gamma) \|x^{(1)} - x^{(0)}\|_\infty$$
$$\leqslant \gamma^K/(1-\gamma) \|x^{(1)} - x^{(0)}\|_\infty$$
$$\leqslant \epsilon \qquad \forall m, k \text{ s.t. } m \geqslant k \geqslant K$$

Hence, $\{x^{(k)}\}$ is convergent for all $x^{(0)} \in D$. Since $\{x^{(k)}\}$ is in D and the set D is closed,

$$x_* := \lim_{k\to\infty} x^{(k)} \in D.$$

Now taking the limit of $x^{(k+1)} = g(x^{(k)})$,

$$x_* = \lim_{k\to\infty} x^{(k+1)} = \lim_{k\to\infty} g(x^{(k)})$$
$$= g(\lim_{k\to\infty} x^{(k)}) = g(x_*).$$

This shows the existence.

As for the uniqueness, suppose $\tilde{x}$ is also a fixed point of g in D. But then

$$\|x_* - \tilde{x}\|_\infty = \|g(x_*) - g(\tilde{x})\|_\infty \leqslant \gamma \|x_* - \tilde{x}\|_\infty$$
$$\Rightarrow \underbrace{(1-\gamma)}_{>0} \|x_* - \tilde{x}\|_\infty \leqslant 0 \Rightarrow x_* = \tilde{x}.$$

(ii) Already proven in part (i).

A Convenient Test to Check Convergence of Simultaneous Iteration

$\underline{g} : \mathbb{R}^n \to \mathbb{R}^n$, continuous on an open subset $D \subset \mathbb{R}^n$

$\dfrac{\partial g_i(x)}{\partial x_j}$ exists at $x = \tilde{x} \in D$ for $i, j = 1, \ldots, n$

The Jacobian matrix $J_{\underline{g}}(x)$ at $x = \tilde{x}$

$$\left[J_{\underline{g}}(\tilde{x}) \right]_{ij} = \frac{\partial g_i(\tilde{x})}{\partial x_j}$$

Ex

$\underline{g} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\underline{g}(x_1, x_2) = \begin{bmatrix} x_1 - x_1^2 - 3x_1x_2 - x_2^2 \\ x_2 - x_1^2 + x_1x_2 - x_2^2 \end{bmatrix}$$

$$(+) \quad J_{\underline{g}}(x_1, x_2) = \begin{bmatrix} -2x_1 - 3x_2 + 1 & -3x_1 - 2x_2 \\ -2x_1 + x_2 & 1 + x_1 - 2x_2 \end{bmatrix}$$

$$J_{\underline{g}}(1, 1) = \begin{bmatrix} -4 & -5 \\ -1 & 0 \end{bmatrix}$$

THM

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous on D, a closed subset of $\mathbb{R}^n$, with a fixed point $x_* \in D$.

Furthermore, suppose there exist an open neighborhood $N(x_*) \subset D$ of x_* where $\partial g_i(x)/\partial x_j$ is continuous for $i,j = 1, \dots, n$.
Suppose also

$$\| J_g(x_*) \|_\infty < 1.$$

Then there exists a closed ball $\overline{B}_\varepsilon(x_*) \subset N(x_*) \subset D$ such that for all $x^{(0)} \in \overline{B}_\varepsilon(x_*)$ the sequence $\{x^{(k)}\}$, $x^{(k+1)} = g(x^{(k)})$ converges to x_*.

Ex

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) = \begin{bmatrix} x_1^2 + 3x_1x_2 + x_2^2 - 29/400 \\ x_1^2 - x_1x_2 + x_2^2 - 13/400 \end{bmatrix}$$

$x_* = (1/20, 1/5)$ is a root of f.

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \quad g(x_1, x_2) = \begin{bmatrix} x_1 - x_1^2 - 3x_1x_2 - x_2^2 + \frac{29}{400} \\ x_2 - x_1^2 + x_1x_2 - x_2^2 + \frac{13}{400} \end{bmatrix}$$

$J_g(x_1, x_2)$ as in (+) on page 1

$$J_g(1/20, 1/5) = \begin{bmatrix} 7/20 & -11/20 \\ 1/10 & 13/20 \end{bmatrix} \quad \| J_g(1/20, 1/5) \|_\infty = \frac{18}{20}$$

By thm above, the sequence $\{x^{(k)}\}$

$$x_1^{(k+1)} = x_1^{(k)} - \left[x_1^{(k)}\right]^2 - 3x_1^{(k)}x_2^{(k)} - \left[x_2^{(k)}\right]^2 - \frac{29}{400}$$

$$x_2^{(k+1)} = x_2^{(k)} - \left[x_1^{(k)}\right]^2 + x_1^{(k)}x_2^{(k)} - \left[x_2^{(k)}\right]^2 + \frac{13}{400}$$

converges to $(1/20, 1/5)$ for all $x^{(0)}$ sufficiently close to $(1/20, 1/5)$.

Proof

Since $\partial g_i(x)/\partial x_j$ are continuous on $N(x_*)$ there exist a closed ball $\overline{B}_\varepsilon(x_*)$ such that

$$\| J_g(x) \|_\infty \leq U \quad \forall x \in \overline{B}_\varepsilon(x_*)$$

where U is such that $\|J_g(x_*)\|_\infty < U < 1$.

We first show g is a contraction on $\overline{B}_\varepsilon(x_*)$.
To this end, define $\phi : [0,1] \to \mathbb{R}$

$$\phi(t) := g_i(t\tilde{x} + (1-t)\tilde{y})$$

for any $\tilde{x}, \tilde{y} \in \overline{B}_\varepsilon(x_*)$. By MVT $\exists \eta \in (0,1)$

$$\phi(1) - \phi(0) = \phi'(\eta)$$

$$\Longleftrightarrow \quad g_i(\tilde{x}) - g_i(\tilde{y}) = \sum_{j=1}^{n} (\tilde{x}_j - \tilde{y}_j) \frac{\partial g_i(\eta\tilde{x} + (1-\eta)\tilde{y})}{\partial x_j}$$

But then

$$(*) \ |g_i(\tilde{x}) - g_i(\tilde{y})| \leq \|\tilde{x} - \tilde{y}\|_\infty \sum_{j=1}^{n} \left| \frac{\partial g_i(\eta\tilde{x} + (1-\eta)\tilde{y})}{\partial x_j} \right|$$

$$\leq \|\tilde{x} - \tilde{y}\|_\infty \, \|J_g(\eta\tilde{x} + (1-\eta)\tilde{y})\|_\infty$$

$$\leq U \|\tilde{x} - \tilde{y}\|_\infty \Rightarrow \|g(\tilde{x}) - g(\tilde{y})\|_\infty \leq U \|\tilde{x} - \tilde{y}\|_\infty$$

where the last inequality is due to $\|J_g(\eta\tilde{x} + (1-\eta)\tilde{y})\|_\infty \leq U$

as $\eta\tilde{x} + (1-\eta)\tilde{y} \in \overline{B}_\varepsilon(x_*)$.

Additionally, $g(\overline{B}_\varepsilon(x_*)) \subseteq \overline{B}_\varepsilon(x_*)$, as it follows from $(*)$ (on page 3) that

$$\|g(x) - g(x_*)\|_\infty = \|g(x) - x_*\|_\infty$$
$$\leqslant U\|x - x_*\|_\infty \leqslant \varepsilon$$

for all $x \in \overline{B}_\varepsilon(x_*)$, implying $g(x) \in \overline{B}_\varepsilon(x_*)$ for such x. Hence, the result is now a consequence of the contraction mapping thm. $\square$