CH5 − P1

Eigenvalues and eigenvectors of a symmetric matrix

Given $A \in \mathbb{R}^{n \times n}$, if

$$Av = \lambda v \qquad \exists v \in \mathbb{C}^n,\ v \neq 0$$

(i) $\lambda \in \mathbb{C}$ is an eigenvalue of A

(ii) $v \in \mathbb{C}^n$ is an eigenvector corresponding to λ.

First order (linear) differential equations with constant coefficients

$$(+) \quad \frac{dx}{dt} = \underbrace{A}_{n \times n} x$$

$n \times n$ constant (independent of t)

$x : \mathbb{R} \longrightarrow \mathbb{C}^n$ (solution sought)

If A is diagonal,

$$\frac{dx_1}{dt} = a_{11} x_1 \qquad \Rightarrow x_1(t) = c_1 e^{a_{11} t}$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{nn} x_n \qquad \Rightarrow x_n(t) = c_n e^{a_{nn} t}$$

$\exists c_1, \dots, c_n$ (scalars)

Now suppose

$$A = V D V^{-1}$$

for some invertible $V \in \mathbb{C}^{n \times n}$

diagonal $D \in \mathbb{C}^{n \times n}$

(+) can be rewritten as

$$\frac{d\underline{y}}{dt} = D\underline{y}$$

with $\underline{y} := V^{-1}x$.

Hence,

$$\underline{y}(t) = \begin{bmatrix} c_1 e^{d_{11}t} \\ \vdots \\ c_n e^{d_{nn}t} \end{bmatrix}$$

$$x(t) = V\underline{y}(t)$$

$$= c_1 e^{d_{11}t} v_1 + \cdots + c_n e^{d_{nn}t} v_n$$

Ex

$$\frac{dx(t)}{dt} = \underbrace{\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}}_{A} x(t)$$

eigenvalues of A

$\lambda_1 = 4 \quad \lambda_2 = -2$

corresponding eigenvector

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$x(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for some constants c_1, c_2.

From here on assume A is symmetric, that is $A^T = A$.

Notation:

$$\mathbb{R}^{n\times n}_{sym} := \{A \in \mathbb{R}^{n\times n} \mid A^T = A\}$$

Recall for $A \in \mathbb{R}^{n\times n}$,

* its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are all real,
* the corresponding eigenvectors $v_1, v_2, \dots, v_n$ can be chosen so that $$\{v_1, \dots, v_n\}$$ is orthonormal

Power Iteration

Generates a sequence $\{q^{(k)}\}$ in $\mathbb{R}^n$ s.t.

$$q^{(k+1)} := \frac{A q^{(k)}}{\| A q^{(k)} \|}$$

let us use 2-norm, but other norms are also fine

(for a given $q^{(0)} \neq 0$.)

What happens to $q^{(k)}$ as $k \to \infty$?

$$q^{(k)} = s_k A^k q^{(0)}$$

where $s_k := 1 / \| A^k q^{(0)} \|$

Eigenvalues in sorted order, that is

$$|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots \leqslant |\lambda_{n-1}| \leqslant |\lambda_n|$$

Expand $q^{(0)}$ in terms of $v_1, \cdots, v_{n-1}, v_n$

$$q^{(0)} = \alpha_n v_n + \alpha_{n-1} v_{n-1} + \cdots + \alpha_1 v_1$$

$$A q^{(0)} = \alpha_n \lambda_n v_n + \alpha_{n-1} \lambda_{n-1} v_{n-1} + \cdots + \alpha_1 \lambda_1 v_1$$

$$\vdots$$

$$A^k q^{(0)} = \alpha_n \lambda_n^k v_n + \alpha_{n-1} \lambda_{n-1}^k v_{n-1} + \cdots + \alpha_1 \lambda_1^k v_1$$

Hence,

$$(++)\ q^{(k)} = s_k \lambda_n^k \left\{ \alpha_n v_n + \alpha_{n-1} \left(\frac{\lambda_{n-1}}{\lambda_n} \right)^k v_{n-1} + \cdots + \alpha_1 \left(\frac{\lambda_1}{\lambda_n} \right)^k v_1 \right\}$$

Assumptions: $\alpha_n \neq 0$ and $|\lambda_{n-1}| < |\lambda_n|$

Some observations

① $|s_k \lambda_n^k| < \dfrac{1}{|\alpha_n|}$ $\quad k = 1, 2, 3, \cdots$

② $\| q^{(k)} - s_k \lambda_n^k \alpha_n v_n \| \to 0$ as $k \to \infty$

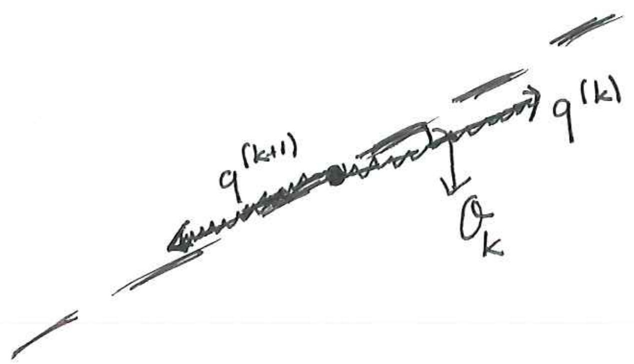
③ $|s_k \lambda_n^k \alpha_n| \to 1$ as $k \to \infty$

Conclusion (Convergence)

There exists a sequence $\{c_k\}$ in $\mathbb{R}$ s.t.

(i) $\lim_{k\to\infty} |c_k| = 1$, and

(ii) $\lim_{k\to\infty} \| q^{(k)} - c_k v_n \|_2 = 0$.



$\theta_k \to 0$ as $k \to \infty$

Order of Convergence

It follows from (++) on page 4 that

$$\| q^{(k)} - s_k \lambda_n^k \alpha_n v_n \| =$$

$$s_k \lambda_n^k \alpha_n \left\{ \frac{\alpha_{n-1}}{\alpha_n} \left(\frac{\lambda_{n-1}}{\lambda_n} \right)^k v_{n-1} + \cdots + \frac{\alpha_1}{\alpha_n} \left(\frac{\lambda_1}{\lambda_n} \right)^k v_1 \right\}$$

Additional assumptions: $\alpha_{n-1} \neq 0$ and $|\lambda_{n-2}| < |\lambda_{n-1}|$

$$\lim_{k\to\infty} \frac{\| q^{(k+1)} - s_{k+1} \lambda_n^{k+1} \alpha_n v_n \|}{\| q^{(k)} - s_k \lambda_n^k \alpha_n v_n \|} = \lim_{k\to\infty} \frac{\left\| \frac{\alpha_{n-1}}{\alpha_n} \left(\frac{\lambda_{n-1}}{\lambda_n} \right)^{k+1} v_{n-1} + \cdots + \frac{\alpha_1}{\alpha_n} \left(\frac{\lambda_1}{\lambda_n} \right)^{k+1} v_1 \right\|}{\left\| \frac{\alpha_{n-1}}{\alpha_n} \left(\frac{\lambda_{n-1}}{\lambda_n} \right)^k v_{n-1} + \cdots + \frac{\alpha_1}{\alpha_n} \left(\frac{\lambda_1}{\lambda_n} \right)^k v_1 \right\|}$$

$$= \left| \frac{\lambda_{n-1}}{\lambda_n} \right|$$

Ex

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\lambda_1 = -2 \quad \lambda_2 = 4 \qquad\qquad \lambda_1 = 1 \quad \lambda_2 = 3$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad\qquad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Suppose $q^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\{q^{(k)}\}$ s.t.

$$q^{(k+1)} = \frac{A q^{(k)}}{\| A q^{(k)} \|}$$

$\{p^{(k)}\}$ s.t.

$$p^{(k+1)} = \frac{B p^{(k)}}{\| B p^{(k)} \|}$$

(i)

$\exists \{c_k\}$ s.t.

$|c_k| \to 1$

and

$\left\| q^{(k)} - c_k \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\| \to 0$

$\exists \{d_k\}$ s.t.

$|d_k| \to 1$

and

$\left\| p^{(k)} - d_k \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\| \to 0$

(ii)

$$\lim_{k\to\infty} \frac{\left\| q^{(k+1)} - c_{k+1} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\|}{\left\| q^{(k)} - c_k \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\|} = \left| \frac{-2}{4} \right| = 1/2$$

$$\lim_{k\to\infty} \frac{\left\| p^{(k+1)} - d_{k+1} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\|}{\left\| p^{(k)} - d_k \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\|} = \frac{1}{3}$$

Jacobi's Method

To compute all eigenvalues of $A \in \mathbb{R}^{n\times n}_{sym}$

Algorithm

Repeat the following (letting $A^{(0)} := A$) for $k = 0, 1, 2, \ldots$ until $\sum_{\substack{j,\ell=1 \\ j\neq \ell}}^{n} \left[a_{j\ell}^{(k)}\right]^2 < \varepsilon$ — prescribed tolerance

Find $p, q \in \{1, 2, \ldots, n\}$ but $p \neq q$ s.t.

$$|a_{pq}^{(k)}| = \max_{\substack{j,\ell=1,\ldots,n \\ j\neq \ell}} |a_{j\ell}^{(k)}|$$

$$A^{(k+1)} \leftarrow R^{(pq)}(\theta_k)^T A^{(k)} R^{(pq)}(\theta_k)$$

end

$a_{j\ell}^{(k)}$ — (j,ℓ) entry of $A^{(k)}$

$R^{(pq)}(\theta_k)$ — orthogonal rotation matrix $(n\times n)$ s.t. $a_{pq}^{(k+1)} = 0$.

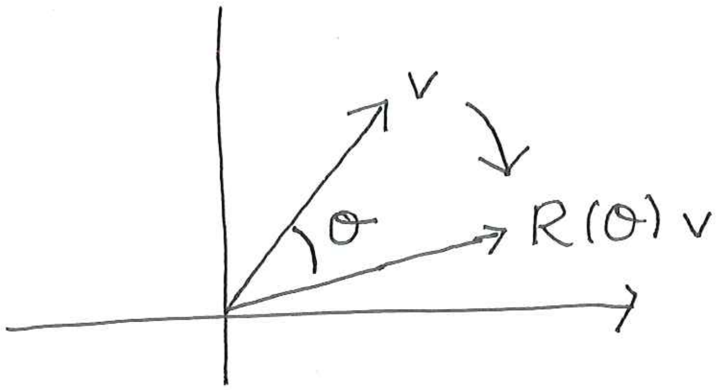
Remarks

(1) $A^{(k)}$ is symmetric $\forall k$.

(2) $A, A^{(1)}, \ldots, A^{(k)}$ $\forall k$ are similar with the same eigenvalues

Rotation Matrices

Rotation in $\mathbb{R}^2$



Rotate v by an angle of θ in clock-wise direction

$$v \longmapsto T_\theta(v)$$

is linear.

$$T_\theta(v) = R(\theta)\,v \qquad \exists\, R(\theta) \in \mathbb{R}^{2\times 2}$$

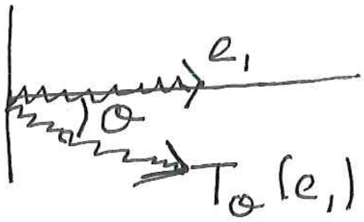
Due to linearity

$$T_\theta(v_1 e_1 + v_2 e_2) = v_1 T_\theta(e_1) + v_2 T_\theta(e_2)$$

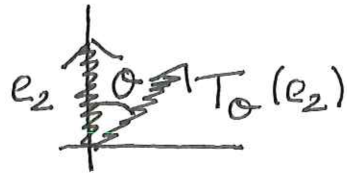
$$= \underbrace{\left[T_\theta(e_1)\ T_\theta(e_2)\right]}_{R(\theta)} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

where

$$T_\theta(e_1) = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$



$$T_\theta(e_2) = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$$



Hence,

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

e.g. $R(\pi/6) = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$ rotates by $\pi/6$ in clock-wise direction

Remark: $R(\theta)$ is orthogonal, that is

$$R(\theta)R(\theta)^T = R(\theta)^T R(\theta) = I_2$$

Planar Rotation in $\mathbb{R}^n$ on the $x_p x_q$-plane

$$
R^{(pq)}(\theta) = \begin{array}{l}
\left[\begin{array}{ccccc}
I & & & & \\
 & c & & s & \\
 & & I & & \\
 & -s & & c & \\
 & & & & I
\end{array}\right]
\begin{array}{l}
\\
\rightarrow p\text{th row} \\
\\
\rightarrow q\text{th row} \\
\\
\end{array}
\end{array}
$$

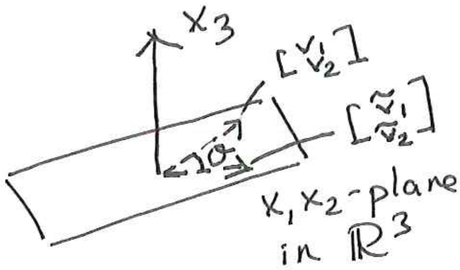
$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
p\text{th column} & & q\text{th column}
\end{array}
$$

$c := \cos(\theta)$
$s := \sin(\theta)$

void entries are 0

$$\tilde{v} := R^{(pq)}(\theta)\, v$$

$$
\begin{cases}
\tilde{v}_j = v_j & j = 1, \ldots, n,\ j \neq p,\ j \neq q \\
\tilde{v}_p = c v_p + s v_q & \\
\tilde{v}_q = -s v_p + c v_q &
\end{cases}
$$



Remark

$R^{(pq)}(\theta)$ is orthogonal, that is

$$I_n = R^{(pq)}(\theta)^T R^{(pq)}(\theta) = R^{(pq)}(\theta)\, R^{(pq)}(\theta)^T$$

Determining θ s.t. (p,q) entry of

$$B := R^{(pq)}(\theta)^T \underbrace{A\, R^{(pq)}}_{\tilde{A}}$$

is zero.

pth ~~row~~ column of $\tilde{A}$

$$\tilde{a}_{jp} = c\, a_{jp} \cancel{+} - s\, a_{jq} \quad (j = 1, \ldots, n)$$

qth column of $\tilde{A}$

$$\tilde{a}_{jq} = s\, a_{jp} + c\, a_{jq} \quad (j = 1, \ldots, n)$$

pth row of B

$$b_{pj} = c\,\tilde{a}_{pj} - s\,\tilde{a}_{qj} \quad (j=1,\ldots,n)$$

q th row of B

$$b_{qj} = s\,\tilde{a}_{pj} + c\,\tilde{a}_{qj} \quad (j=1,\ldots,n)$$

Only pth and qth rows and columns of B are different compared with A.

Need to determine c and s (and θ).
Should exploit $b_{pq}=0$.

$$\begin{aligned} 0 = b_{pq} &= c\,\tilde{a}_{pq} - s\,\tilde{a}_{qq} \\ &= c\{s\,a_{pp} + c\,a_{pq}\} \\ &\quad - s\{s\,a_{qp} + c\,a_{qq}\} \\ &= cs(a_{pp}-a_{qq}) + (c^2-s^2)\,a_{pq} \end{aligned}$$

Recall $a_{pq}=a_{qp}$

$$\Longrightarrow \underbrace{\frac{2cs}{c^2-s^2}}_{\left(\frac{2\cos\theta\,\sin\theta}{\cos^2\theta-\sin^2\theta}=\tan 2\theta\right)} = \frac{2a_{pq}}{a_{qq}-a_{pp}}$$

$$\Longrightarrow \underbrace{\theta}_{\in[-\pi/4,\,\pi/4]} = \frac{1}{2}\arctan \underbrace{\frac{2a_{pq}}{a_{qq}-a_{pp}}}_{\in[-\pi/2,\,\pi/2]}$$

(if $a_{pp}=a_{qq}$ $\theta=\pi/4$ or $\theta=-\pi/4$ depending on sign of a_{pq})

$$c = \cos\theta$$
$$s = \sin\theta$$

Convergence

The Frobenius norm of $A \in \mathbb{R}^{n\times n}$

$$\|A\|_F := \sqrt{\sum_{j,k=1}^{n} a_{jk}^2}$$

is invariant under orthogonal similarity transformations.

THM

Let $A \in \mathbb{R}^{n\times n}$ and $B = Q^T A Q \in \mathbb{R}^{n\times n}$ for some orthogonal matrix $Q \in \mathbb{R}^{n\times n}$. Then

$$\|A\|_F = \|B\|_F .$$

Proof

$$\begin{aligned}
\|B\|_F^2 &= \text{Trace}\,(B^T B) \\
&= \text{Trace}\,(Q^T A^T Q \cdot Q^T A Q) \\
\left(\text{as } QQ^T = I\right) \quad &= \text{Trace}\,(Q^T A^T A Q) \\
\left(\text{as Traces of 2 similar matrices are same}\right) \quad &= \text{Trace}\,(A^T A) = \|A\|_F^2 \\
&\Longrightarrow \|A\|_F = \|B\|_F . \qquad \square
\end{aligned}$$

THM

The sequence $\{A^{(k)}\}$ of matrices by Jacobi's method is such that

$$\lim_{k\to\infty} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left[a_{ij}^{(k)}\right]^2 = 0.$$

Proof

Define

$$S(A^{(k)}) := \sum_{i,j=1}^{n} \left[a_{ij}^{(k)}\right]^2 = \|A^{(k)}\|_F^2$$

$$D(A^{(k)}) := \sum_{i=1}^{n} \left[a_{ii}^{(k)}\right]^2$$

$$L(A^{(k)}) := \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left[a_{ij}^{(k)}\right]^2$$

so that $S(A^{(k)}) = D(A^{(k)}) + L(A^{(k)})$

as well as $S(A^{(k)}) = S(A^{(k+1)})$.

Recall

$$\begin{bmatrix} a_{pp}^{(k+1)} & a_{pq}^{(k+1)} \\ a_{qp}^{(k+1)} & a_{qq}^{(k+1)} \end{bmatrix} = \cancel{R^{(pq)}(\theta)} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}^T \begin{bmatrix} a_{pp}^{(k)} & a_{pq}^{(k)} \\ a_{qp}^{(k)} & a_{qq}^{(k)} \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}.$$

Hence, by previous thm

$$\left[a_{pp}^{(k)}\right]^2 + 2\left[a_{pq}^{(k)}\right]^2 + \left[a_{qq}^{(k)}\right]^2 = \left[a_{pp}^{(k+1)}\right]^2 + 2\left[a_{pq}^{(k+1)}\right]^2 + \left[a_{qq}^{(k+1)}\right]^2$$

$$= \left[a_{pp}^{(k+1)}\right]^2 + \left[a_{qq}^{(k+1)}\right]^2.$$

As all diagonal entries of $A^{(k)}$ and $A^{(k+1)}$ are the same, except (p,p) and (q,q) entry,

$$D(A^{(k+1)}) = D(A^{(k)}) + 2\left[a_{pq}^{(k)}\right]^2$$

$\boxed{\text{since } S(A^{(k+1)}) = S(A^{(k)})}$

$$\Longrightarrow L(A^{(k+1)}) = L(A^{(k)}) - 2\left[a_{pq}^{(k)}\right]^2.$$

Now $L(A^{(k)}) \leqslant (n^2 - n)\left[a_{pq}^{(k)}\right]^2 \Rightarrow -2\left[a_{pq}^{(k)}\right]^2 \leqslant \frac{-2}{n^2-n} L(A^{(k)})$,

so

$$L(A^{(k+1)}) \leqslant \underbrace{\left(1 - \frac{2}{n^2-n}\right)}_{\in [0,1)} L(A^{(k)})$$

implying

$$L(A^{(k)}) \leqslant \left(1-\frac{2}{n^2-n}\right)^k L(A)$$

so that $\lim_{k\to\infty} L(A^{(k)}) = 0$ as desired. $\square$

Summary

1. $A, A^{(k)}$ $\forall k$ have the same eigenvalues.
2. $A^{(k)}$ becomes diagonal as $k \to \infty$.
3. diagonal entries of $A^{(k)}$ become eigenvalues of A as $k \to \infty$.