CH 6 - P1

Lagrange Interpolation

Given

$$(x_0, y_0),\ (x_1, y_1),\ \dots,\ (x_n, y_n)$$

with

$$x_0, \dots, x_n \in \mathbb{R} \quad \text{s.t.} \quad x_j \neq x_k \text{ for } j \neq k,$$

$$y_0, \dots, y_n \in \mathbb{R},$$

find a polynomial $p_n \in P_n$ satisfying

$$p_n(x_j) = y_j \qquad j = 0, 1, \dots, n.$$

(P_n denotes the set of polynomials $p: \mathbb{R} \to \mathbb{R}$ of degree at most n.)

Would like to find $p_n \in P_n$ of the form

$$(+) \quad p_n(x) = \sum_{k=0}^{n} L_k(x)\, y_k$$

where $\underbrace{L_k(x)}_{\in P_n}$ is to be chosen such that

$$L_k(x_j) = \begin{cases} 1 & j = k \\ 0 & j \in \{0, 1, \dots, n\},\ j \neq k \end{cases}$$

Hence,

$$L_k(x) = C_k \prod_{\substack{j=0 \\ j\neq k}}^{n} (x - x_j)$$

where C_k is such that $L_k(x_k) = 1$, that is

$$L_k(x) = \left\{ \prod_{\substack{j=0 \\ j\neq k}}^{n} (x - x_j) \right\} \Big/ \left\{ \prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j) \right\}.$$

<u>THM</u> (Lagrange Interpolation Thm)
There exists a unique polynomial $p_n \in P_n$ satisfying $p_n(x_j) = y_j$ for $j = 0, 1, \dots, n$.

<u>Proof</u>

Existence follows from (+) on page ①.
As for the uniqueness suppose there is another polynomial $q_n \in P_n$ such that $q_n(x_j) = y_j$. But then $h_n := p_n - q_n \in P_n$ has $(n+1)$ roots, namely $x_0, \dots, x_n$ as

$$h_n(x_j) = p_n(x_j) - q_n(x_j) = y_j - y_j = 0.$$

Consequently,

$$h_n(x) \equiv 0 \implies p_n(x) \equiv q_n(x). \qquad \square$$

Ex

Find the unique polynomial $p_2 \in P_2$ through

$(1,1)$, $(2,4)$, $(3,8)$.

$$p_2(x) = L_0(x)\cdot 1 + L_1(x)\cdot 4 + L_2(x)\cdot 8$$

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)} + 4\,\frac{(x-1)(x-3)}{(2-1)(2-3)} + 8\,\frac{(x-1)(x-2)}{(3-1)(3-2)}$$

$$= \frac{1}{2}(x^2-5x+6) - 4(x^2-4x+3) + 4(x^2-3x+2)$$

$$= \frac{1}{2}x^2 + \frac{3}{2}x - 1$$

$$p_n(x) = \sum_{k=0}^{n} L_k(x)\, y_k$$

is called the Lagrange interpolation polynomial of degree n for the set of points $\{(x_j, y_j) \mid j = 0, \ldots, n\}$.

Could also use the Lagrange interpolation to approximate a function $f: \mathbb{R} \to \mathbb{R}$ with a polynomial. Given $x_0, x_1, \ldots, x_n \in \mathbb{R}$ such that $x_j \neq x_\ell$ for $j \neq \ell$

$$p_n(x) = \sum_{k=0}^{n} L_k(x)\, f(x_k)$$

is called the Lagrange interpolation polynomial of degree n for the function f with the interpolation points $x_0, \ldots, x_n$.

Ex

Lagrange interpolation polynomial of degree (at most) P_2 for $f(x) = \cos x$ with interpolation points $0, \pi/6, \pi/3$.

$$p_2(x) = \cos 0 \cdot \frac{(x-\pi/6)(x-\pi/3)}{(0-\pi/6)(0-\pi/3)} + \cos \pi/6 \, \frac{(x-0)(x-\pi/3)}{(\pi/6-0)(\pi/6-\pi/3)}$$

$$+ \cos \pi/3 \, \frac{(x-0)(x-\pi/6)}{(\pi/3-0)(\pi/3-\pi/6)}$$

$$= \frac{18}{\pi^2}\left(x^2 - \frac{\pi}{2}x + \frac{\pi^2}{18}\right) - \frac{18\sqrt{3}}{\pi^2}\left(x^2 - \frac{\pi}{3}x\right)$$

$$+ \frac{9}{\pi^2}\left(x^2 - \frac{\pi}{6}x\right)$$

$$= \frac{27-18\sqrt{3}}{\pi^2} x^2 + \frac{(6\sqrt{3}-21/2)}{\pi} x + 1$$

THM (Interpolation Error)

Let $f : \mathbb{R} \to \mathbb{R}$ be such that $(n+1)$st derivative of f exists at all $x \in [a,b]$, and $p_n \in \mathbb{P}_n$ be the Lagrange interpolation polynomial with distinct interpolation points $x_0, x_1, \ldots, x_n \in [a,b]$. Then for all $x \in [a,b]$, we have

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x)$$

for some $\xi \in (a,b)$, where

$$\pi_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n).$$

Proof

If $x = x_j$ $\exists j \in \{0, 1, \ldots, n\}$, nothing to prove as $f(x_j) = p_n(x_j)$ and $\Pi_{n+1}(x_j) = 0$. Assume $x \neq x_j$, and define

$$\varphi : [a, b] \to \mathbb{R},$$
$$\varphi(t) := f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\Pi_{n+1}(x)} \Pi_{n+1}(t).$$

Now, φ has $n+2$ roots on $[a, b]$, namely $x_0, x_1, \ldots, x_n$ and x. By MVT φ' has $(n+1)$ roots, φ'' has n roots, $\varphi^{(n+1)}$ has one root on (a, b) say ϵ. But then

$$0 = \varphi^{(n+1)}(\epsilon) = f^{(n+1)}(\epsilon) - p_n(\epsilon) - \frac{f(x) - p_n(x)}{\Pi_{n+1}(x)} \Pi_{n+1}^{(n+1)}(\epsilon)$$
$$= f^{(n+1)}(\epsilon) - \frac{f(x) - p_n(x)}{\Pi_{n+1}(x)} (n+1)!$$
$$\Longrightarrow f(x) - p_n(x) = \frac{f^{(n+1)}(\epsilon)}{(n+1)!} \Pi_{n+1}(x)$$

as desired. □

Corrollary

With the assumptions of the previous thm, and additionally if $f^{(n+1)}(x)$ is continuous on $[a, b]$, for all $x \in [a, b]$ we have

$$|f(x) - p_n(x)| \leqslant M_{n+1} |\Pi_{n+1}(x)| / (n+1)!$$

where M_{n+1} is such that $|f^{(n+1)}(\tilde{x})| \leqslant M_{n+1}$ $\forall \tilde{x} \in [a, b]$.

Ex

Consider

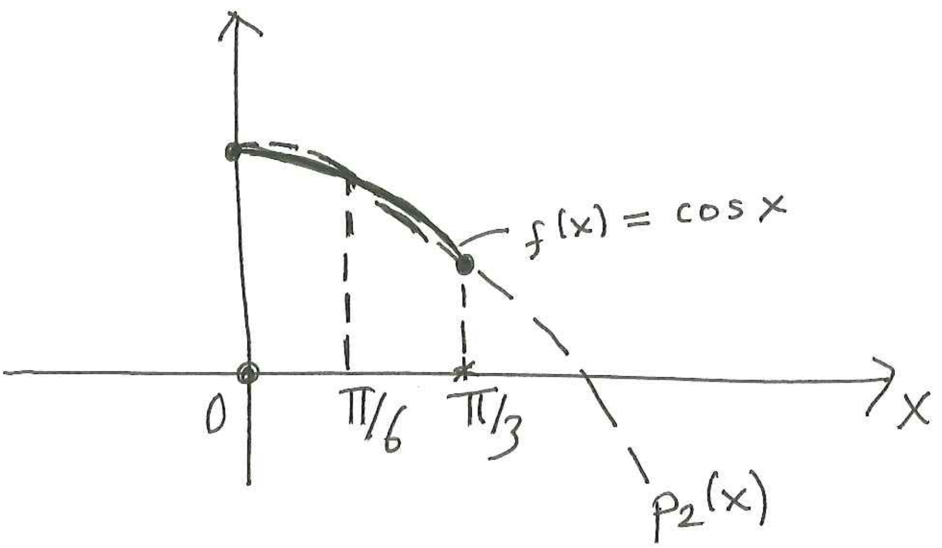
$$f(x) = \cos x \quad \text{and} \quad P_2(x) = \frac{27-18\sqrt{3}}{\pi^2}x^2 + \frac{6\sqrt{3}-21/2}{\pi} + 1$$

(the interpolation polynomial with interpolation points $0, \pi/6, \pi/3$); see the example on page 4.

$$|f(x) - p_2(x)| \leqslant \frac{1}{2}\left|(x-0)(x-\pi/6)(x-\pi/3)\right|$$

$$\leqslant \frac{1}{2}\left(\frac{\pi}{12}\right)^2 \cdot \frac{\pi}{3} < 1/12$$

for all $x \in [0, \pi/3]$.



CH 6 – P2

Hermite Interpolation

Given

$$(x_j, y_j, z_j) \qquad j = 0, \dots, n$$

where

$x_0, \dots, x_n \in \mathbb{R}$ are distinct,

$y_0, \dots, y_n, z_0, \dots, z_n \in \mathbb{R}$,

find a polynomial $p \in P_{2n+1}$ such that

$$(+) \quad p(x_j) = y_j \quad \text{and} \quad p'(x_j) = z_j$$

for $j = 0, 1, \dots, n$.

$$(*) \quad p(x) = \sum_{k=0}^{n} H_k(x)\, y_k + K_k(x)\, z_k$$

with

$$H_k(x) = [L_k(x)]^2 \left(1 - 2L'(x_k)(x - x_k)\right)$$

$$K_k(x) = [L_k(x)]^2 (x - x_k)$$

Observe

(1) $H_k, K_k \in P_{2n+1} \implies \overset{\text{as in } (*)}{p} \in P_{2n+1}$

(2) $$H_k(x_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k, \ j \in \{0, 1, \dots, n\} \end{cases}$$

$$H_k'(x_j) = 0 \qquad j = 0, 1, \dots, n$$

(3) $K_k(x_j) = 0 \qquad j = 0, 1, \ldots, n$

$$K'_k(x_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k,\ j \in \{0, 1, \ldots, n\} \end{cases}$$

Consequently,

p as in $(*)$ satisfies $(+)$.

THM

There exists a unique polynomial $p \in P_{2n+1}$ such that

$$p(x_j) = y_j, \quad p'(x_j) = z_j \qquad j = 0, 1, \ldots, n.$$

Proof

Existence is shown by $(*)$.

As for the uniqueness assume $q \in P_{2n+1}$ is such that

$$q(x_j) = y_j, \quad q'(x_j) = z_j \qquad j = 0, 1, \ldots, n.$$

Define $h := p - q$ ($\in P_{2n+1}$). We ~~have~~ have $h(x_j) = 0$ for $j = 0, 1, \ldots, n$, so by MVT

$$h'(\eta_j) = 0 \qquad \exists \eta_j \in (x_j, x_{j+1})$$

for $j = 0, 1, \ldots, n-1$.

But this means h' ($\in P_{2n}$) has $2n+1$ distinct roots, ~~By~~ so $h'(x) \equiv 0$ implying $h(x)$ is a constant function. Since $h(x_j) = 0$ $j = 0, 1, \ldots, n$, we must have $h(x) \equiv 0$ and $p(x) \equiv q(x)$ proving the uniqueness. $\square$

Hermite interpolation polynomial of degree $2n+1$ for the set $\{(x_j, y_j, z_j) \mid j = 0, 1, \ldots, n\}$

$$p(x) = \sum_{k=0}^{n} H_k(x)\, y_k + K_k(x)\, z_k$$

Hermite interpolation polynomial of degree $2n+1$ for the function $f: \mathbb{R} \to \mathbb{R}$ with interpolation points $x_0, x_1, \ldots, x_n$.

$$p(x) = \sum_{k=0}^{n} H_k(x)\, f(x_k) + K_k(x)\, f'(x_k)$$

(satisfies (1) $p(x_k) = f(x_k)$ (2) $p'(x_k) = f'(x_k)$ for $k = 0, 1, \ldots, n$.)

Ex

Find the Hermite interpolation polynomial of degree 3 for $f(x) = \cos x$ with interpolation points $x_0 = 0$, $x_1 = \pi/2$.

$$p \in P_3 \qquad p(0) = 1 \qquad p(\pi/2) = 0$$
$$p'(0) = 0 \qquad p'(\pi/2) = -1$$

$$p(x) = H_0(x) \cdot 1 + K_1(x) \cdot (-1)$$

$$H_0(x) = \left[\frac{(x - \pi/2)}{(0 - \pi/2)}\right]^2 \left(1 - \left(-\frac{2}{\pi}\right)(x - 0)\right)$$

$$= \frac{4}{\pi^2} \underbrace{(x - \pi/2)^2}_{x^2 - \pi x + \pi^2/4} \left(1 + \frac{2}{\pi} x\right) = \frac{8}{\pi^3} x^3 - \frac{4}{\pi^2} x^2 - \frac{2}{\pi} x + 1$$

$$K_0(x) = \left[\frac{x - \pi/2}{0 - \pi/2}\right]^2 (x - 0) = \frac{4}{\pi^2} x^3 - \frac{4}{\pi} x^2 + x$$

Hence,

$$p(x) = \left(\frac{8}{\pi^3}x^3 - \frac{4}{\pi^2}x^2 - \frac{2}{\pi}x + 1\right) - \left(\frac{4}{\pi^2}x^3 - \frac{4}{\pi}x^2 + x\right)$$

$$= \left(\frac{8-4\pi}{\pi^3}\right)x^3 + \left(\frac{4\pi-4}{\pi^2}\right)x^2 - \left(\frac{2+\pi}{\pi}\right)x + 1.$$

THM

Let $f : \mathbb{R} \to \mathbb{R}$ be such that its $(2n+2)$th derivative exists on $[a, b]$, and p be the Hermite interpolation polynomial of degree $2n+1$ with interpolation points $x_0, x_1, \dots, x_n$. Then for every $x \in [a, b]$, we have

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\,\pi_{n+1}(x)^2$$

for some $\xi \in (a, b)$ where

$$\pi_{n+1}(x) = (x-x_0)(x-x_1)\dots(x-x_n).$$

Proof is similar to the one for Lagrange interpolation polynomial but on

$$\varphi(t) := f(t) - p(t) - \frac{f(x)-p(x)}{\pi_{n+1}(x)^2}\,\pi_{n+1}(t)^2.$$

Corollary

Let f and p be as in the previous thm, but with the additional assumption that $(2n+2)$th derivatives of f is continuous on $[a, b]$. Then, we have

$$|f(x) - p(x)| \leq \frac{M_{2n+2}}{(2n+2)!}\,|\pi_{n+1}(x)|^2$$

for any M_{2n+2} such that $|f^{(2n+2)}(\tilde{x})| \leq M_{2n+2}$ $\forall \tilde{x} \in [a, b]$.