LECTURE 20

NONLINEAR OPTIMIZATION WITH INEQUALITY CONSTRAINTS (PART I

$$\text{(NIP)} \quad \begin{array}{l} \text{minimize } f(x) \\ x \in \mathbb{R}^n \\ \text{subject to} \\ \quad c_j(x) \geq 0 \quad j = 1, \ldots, m \end{array}$$

CHARAC (For Local Minimizer)

$x_* \in \mathbb{F}$ is a local minimizer

$$\Longrightarrow$$

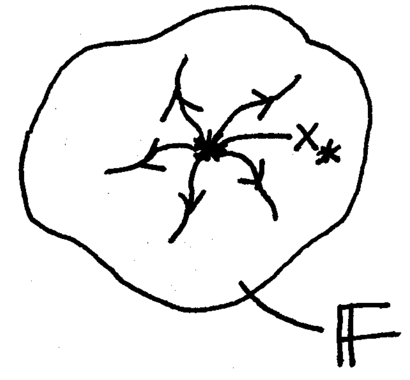
$\nabla f(x_*)^T p \geq 0$ for all $p \in T^0(x_*)$

ALGEBRAIC CHARACTERIZATION FOR THE TANGENT CONE

* If $c_j(x)$ is inactive at x_* (i.e., $c_j(x_*) > 0$) then $c_j(x+p) > 0$ for all $p \in \mathbb{R}^n$ with small nor
* Inactive constraints are irrelevant to a feasible path and the tangent cone.

EXAMPLE

Suppose none of the constraints are active at x_*.



$$T^{o} \text{ at } x_* = \mathbb{R}^n$$

Tangent cone is determined by active constraints.

Suppose $c_j(x) \geqslant 0$ is active at x_*, that is $c_j(x_*) = 0$. For all feasible paths $x(\alpha)$ at x_*

$$c_j(x(\alpha)) \geqslant 0 \quad \text{for all small } \alpha > 0$$

$$\Longrightarrow$$

$$\ell(\alpha) = c_j(x(\alpha)) \text{ is nondecreasing at } \alpha = 0$$

$$\Longrightarrow$$

$$0 \leqslant \ell'(0) = \nabla c_j(x_*)^T \underbrace{x'(0)}_{p \in T^{o} \text{ at } x_*}$$

Let $j_1, j_2, \dots, j_p$ be indices of active constraints at x_*. For all $p \in T^{o}$ at x_*

$$\left.\begin{array}{l} \nabla c_{j_1}(x_*)^T p \geqslant 0 \\ \vdots \\ \nabla c_{j_p}(x_*)^T p \geqslant 0 \end{array}\right\} \Longrightarrow \underbrace{J_a(x_*)p \geqslant 0}_{\text{all components of } J_a(x_*)p \text{ are nonnegative}}$$

where

$$\underbrace{J_a(x_*)}_{\substack{\text{Jacobian of}\\ \text{active constraints}}} = c_a'(x_*) = \begin{bmatrix} \nabla c_{j_1}(x_*)^T \\ \vdots \\ \nabla c_{j_p}(x_*)^T \end{bmatrix}$$

with

$$c_a(x_*) = \begin{bmatrix} c_{j_1}(x_*) \\ \vdots \\ c_{j_p}(x_*) \end{bmatrix}.$$

<u>THM</u> (Algebraic Characterization for T°)

T° at $x_* \subseteq \{p \in \mathbb{R}^n : J_a(x_*)p \geq 0\}$

<u>NOTATION</u>

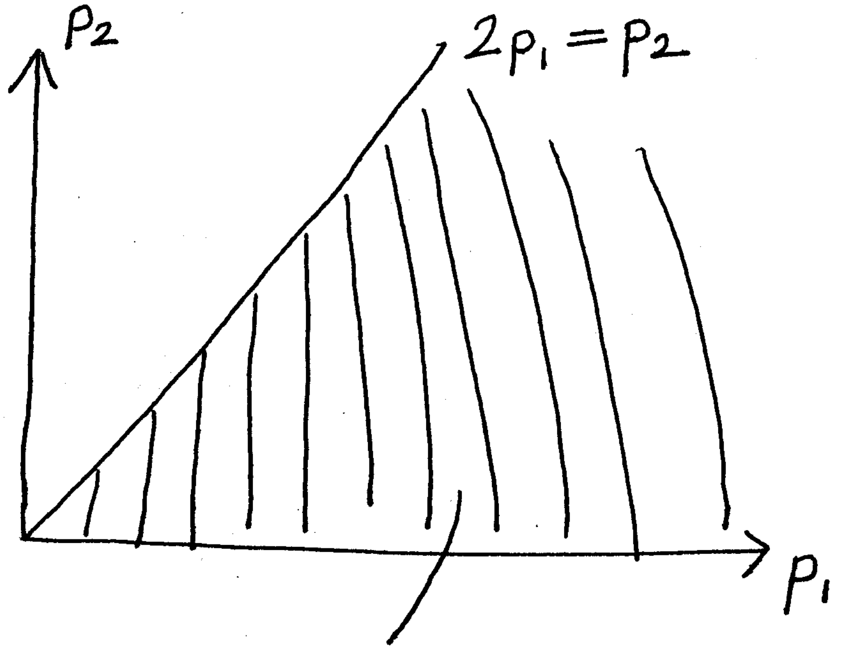
$\mathcal{A}(x_*)$: active set at x_* (set of indices of active constraints at x_*)

<u>EXAMPLE</u>

(1) $-x_1^2 + x_2 \geq 0$ (2) $-x_1^2 + 2x_1 - x_2 \geq 0$

* $\mathcal{A}(0) = \{1, 2\}$
* $J_a(0) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$
* $\{p \in \mathbb{R}^2 : J_a(0)p \geq 0\} =$

$\left\{ \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} : p_2 \geq 0 \text{ and } 2p_1 - p_2 \geq 0 \right\}$



$$\{p \in \mathbb{R}^2 : J_a(0)\,p \geqslant 0\}$$

$$=$$

$$\left\{\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \alpha_1, \alpha_2 \geqslant 0 \right\}$$

$$=$$

T° at 0

In general it is possible that

- T° at $x_* \subset \{p \in \mathbb{R}^n : J_a(x_*)\,p \geqslant 0\}$

But usually

(*) T° at $x_* = \{p \in \mathbb{R}^n : J_a(x_*)\,p \geqslant 0\}$.

REMARKS

* We say that the constraint qualification holds at x_* if (*) holds.

CHARAC 2 (For Local Minimizer)

Suppose that constraint qualification holds at $x_* \in \mathbb{F}$.

x_* is a local minimizer

$$\Longrightarrow$$

$\nabla f(x_*)^T p \geqslant 0$ for all $p \in \mathbb{R}^n$ such that $J_a(x_*) p \geqslant 0$

LINEAR INDEPENDENCE CONSTRAINT QUALIFICATION (LICQ)

Suppose $\mathcal{A}(x_*) = \{j_1, \ldots, j_p\}$ and

$$\{\nabla c_{j_1}(x_*), \ldots, \nabla c_{j_p}(x_*)\}$$

is linearly independent. Then constraint qualification holds at x_*.

EXAMPLE

(1) $\underbrace{-x_1^2 + x_2}_{c_1(x)} \geqslant 0$ (2) $\underbrace{-x_1^2 + 2x_1 - x_2}_{c_2(x)} \geqslant 0$

$$\{\nabla c_1(0), \nabla c_2(0)\} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

is linearly independent

$$\Longrightarrow$$

constraint qualification holds at 0

LEMMA (Farkas)

Following conditions are equivalent.

(1) $\nabla f(x_*)^T p \geq 0$ for all $p \in \mathbb{R}^n$ such that $J_a(x_*)p \geq 0$

(2) $\nabla f(x_*) = J_a(x_*)^T \lambda$ for some $\lambda \geq 0$. (all components of λ are nonnegative)

DEFN (Active Normal Cone)

The set

$$\mathcal{N}_a(x_*) = \{ J_a(x_*)^T \lambda : \lambda \geq 0 \}$$

is called the <u>active normal cone</u>.

Active normal cone is the set consisting of all linear combinations of $\nabla c_{j_1}(x_*), \dots, \nabla c_{j_p}(x_*)$ with positive weights.

$$J_a(x_*)^T \underbrace{\lambda}_{\geq 0} = \left[\nabla c_{j_1}(x_*) \; \dots \; \nabla c_{j_p}(x_*) \right] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{bmatrix}$$

$$= \underbrace{\lambda_1}_{\geq 0} \nabla c_{j_1}(x_*) + \dots + \underbrace{\lambda_p}_{\geq 0} \nabla c_{j_p}(x_*)$$

EXAMPLE

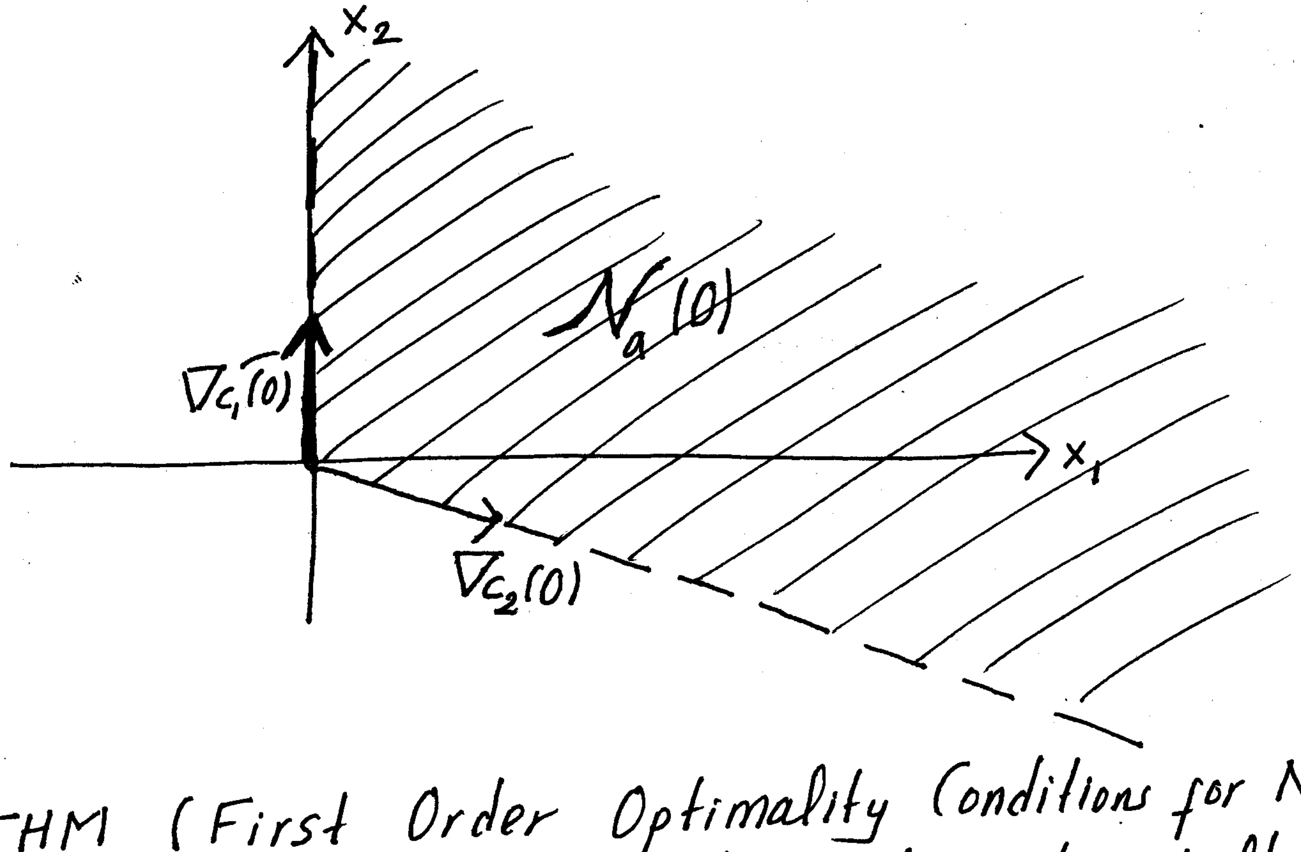
$$(1)\ \underbrace{-x_1^2 + x_2}_{c_1(x)} \geq 0 \qquad (2)\ \underbrace{-x_1^2 + 2x_1 - x_2}_{c_2(x)} \geq 0$$

Both constraints are active at 0 with

$$\nabla c_1(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \nabla c_2(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Active normal cone

$$\mathcal{N}_a(0) = \left\{ \alpha_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} : \alpha_1, \alpha_2 \geq 0 \right\}$$



THM (First Order Optimality Conditions for NIP)
Suppose that constraint qualification holds at x_*. If x_* is a local minimizer, then

(i) $c_j(x_*) \geq 0 \quad j = 1, \dots, m$ and

(ii) $\nabla f(x_*) = J_a(x_*)^T \lambda$ for some $\lambda \geq 0$

(equivalently $\nabla f(x_*) \in \mathcal{N}_a(x_*)$)