LECTURE 24

PRIMAL-DUAL INTERIOR POINT METHODS FOR LINEAR PROGRAMS

Consider LP

$$\text{(LP)}\quad \begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & c^T x \\ \text{subject to} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

$c \in \mathbb{R}^n$
$A \in \mathbb{R}^{m\times n}$
$b \in \mathbb{R}^m$

and its dual

$$\text{(DLP)}\quad \begin{array}{ll} \underset{\pi\in\mathbb{R}^m}{\text{maximize}} & b^T \pi \\ \text{subject to} & \\ & A^T\pi + s = c \\ & s \geq 0 \end{array}$$

Primal-dual interior point methods solve

* (LP) and (DLP), simultaneously
* staying feasible for both problems
* and strictly satisfying constraints $x \geq 0$, $s \geq 0$.

KKT conditions for (LP) and (DLP)

(1) $Ax = b$

(2) $A^T\pi + s = c$

(3) $x_j s_j = 0 \quad j = 1, \dots, n$

(equivalently $XSe = 0$)

(4) $x \geq 0, \quad s \geq 0$

Above $X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & & \\ \vdots & & \ddots & \\ 0 & & & x_n \end{bmatrix}$, $S = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & & \\ \vdots & & \ddots & \\ 0 & & & s_n \end{bmatrix}$

and $e = [1 \; 1 \; \dots \; 1]^T$.

Feasible Regions

FEASIBLE $\mathbb{F}_{P,D} = \{(x,\pi,s) : Ax = b, A^T\pi + s = c, x \geq 0, \; s \geq 0\}$

STRICTLY FEASIBLE $\mathbb{F}^0_{P,D} = \{(x,\pi,s) : Ax = b, \; A^T\pi + s = c, x > 0, \; s > 0\}$

Primal-dual interior point methods search over ~~$\mathbb{F}^0_{P,D}$~~ $(x,\pi,s) \in \mathbb{R}^{2n+m}$ making sure (x,π,s) remains in $\mathbb{F}^0_{P,D}$

EXAMPLE

(LP)
$$\begin{aligned} &\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad x_1 + 2x_2 \\ &\text{subject to} \\ &\quad x_1 + x_2 = 1 \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

(DLP)
$$\begin{aligned} &\underset{\pi \in \mathbb{R},\, s \in \mathbb{R}^2}{\text{maximize}} \quad \pi \\ &\text{subject to} \\ &\quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \pi + s = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &\quad s \geq 0 \end{aligned}$$

$$\boxed{\begin{aligned} c &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ A &= \begin{bmatrix} 1 & 1 \end{bmatrix} \\ b &= 1 \end{aligned}}$$

$$\mathbb{F}_{P,D} = \{ \underbrace{(x, \pi, s)}_{\text{in } \mathbb{R}^5} : x_1 + x_2 = 1,\ \pi + s_1 = 1,\ \pi + s_2 = 2,\ x_1, x_2, s_1, s_2 \geq 0 \}$$

$$\mathbb{F}^0_{P,D} = \{ (x, \pi, s) : x_1 + x_2 = 1,\ \pi + s_1 = 1,\ \pi + s_2 = 2,\ x_1, x_2, s_1, s_2 > 0 \}$$

Central Path

Given $(x, \pi, s) \in \mathbb{F}^0_{P,D}$ except complementarity all KKT conditions are satisfied

Centralization parameter

$$\mu = \frac{1}{n} \sum_{j=1}^{n} x_j s_j$$

To have progress on complementarity and stay away from the boundary of the feasible region we aim to solve

$$\text{(CC)} \quad \begin{aligned} &(1)\ Ax = b \\ &(2)\ A^T\pi + s = c \\ &(3)\ XSe = \sigma\mu \end{aligned}$$

centeralization conditions

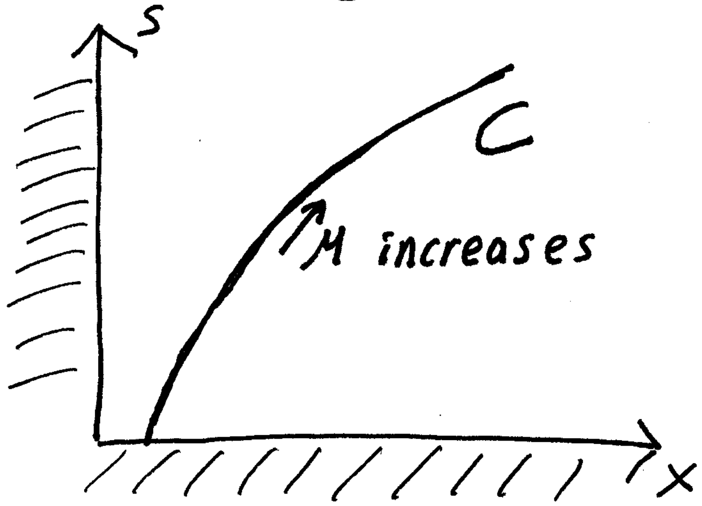
subject to $x \geqslant 0$, $s \geqslant 0$

where $\sigma \in [0, 1]$.

Specifically the set

$$C = \{(x_\mu, \pi_\mu, s_\mu) : Ax_\mu = b,\ A^T\pi_\mu + s_\mu = c,\ (x_\mu)_j (s_\mu)_j = \mu,\ x \geqslant 0,\ s \geqslant 0\}$$

is called the <u>central path</u>.



Newton's Method for Centeralization Conditions

Define $F: \mathbb{R}^{2n+m} \longrightarrow \mathbb{R}^{2n+m}$

$$F(x, \pi, s) = \begin{bmatrix} Ax - b \\ A^T\pi + s - c \\ XSe - \sigma\mu \end{bmatrix}$$

$$F'(x, \pi, s) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix}$$

Newton iteration for given $(x_k, \pi_k, s_k) \in \mathbb{F}_{D,P}$.

(1) Solve

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S_k & 0 & X_k \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta \pi_k \\ \Delta s_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -X_k S_k e + \sigma \mu_k \end{bmatrix}$$

where $X_k = \begin{bmatrix} (x_k)_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & (x_k)_n \end{bmatrix}$, $S_k = \begin{bmatrix} (s_k)_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & (s_k)_n \end{bmatrix}$

and $\mu_k = \frac{1}{n} \sum_{j=1}^{n} x_j s_j$

(2) $(x_{k+1}, \pi_{k+1}, s_{k+1}) = (x_k, \pi_k, s_k) + \alpha_k (\Delta x_k, \Delta \pi_k, \Delta s_k)$
for some $\alpha_k > 0$ s.t. $x_{k+1}, s_{k+1} \geq 0$.

REMARK

The Newton iteration above preserves feasibility, i.e.,

(i) $A x_k = b$ and $A(\Delta x_k) = 0$

$$\Longrightarrow$$

$$A \underbrace{(x_k + \alpha \Delta x_k)}_{x_{k+1}} = 0$$

(ii) $A^T \pi_k + s_k = c$ and $A^T(\Delta \pi_k) + \Delta s_k = 0$

$$\Longrightarrow$$

$$A^T \underbrace{(\pi_k + \alpha_k \Delta \pi_k)}_{\pi_{k+1}} + \underbrace{(s_k + \alpha_k \Delta s_k)}_{s_{k+1}} =$$

EXAMPLE

(LP)
$$\begin{aligned} &\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad x_1 + 2x_2 \\ &\text{subject to} \\ &\quad x_1 + x_2 = 1 \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

(DLP)
$$\begin{aligned} &\underset{\pi \in \mathbb{R},\, s \in \mathbb{R}^2}{\text{maximize}} \quad \pi \\ &\text{subject to} \\ &\quad \pi + s_1 = 1 \\ &\quad \pi + s_2 = 2 \\ &\quad s_1, s_2 \geq 0 \end{aligned}$$

$$F(x, \pi, s) = \begin{bmatrix} x_1 + x_2 - 1 \\ \pi + s_1 - 1 \\ \pi + s_2 - 2 \\ x_1 s_1 - \sigma\mu \\ x_2 s_2 - \sigma\mu \end{bmatrix} \quad \left(F : \mathbb{R}^5 \to \mathbb{R}^5 \right)$$

σ, μ are fixed. $\cancel{\mu = \frac{1}{2}(x_1 s_1 + x_2 s_2)}$

Newton iteration (given $(x_k, \pi_k, s_k) \in \mathbb{F}^{0}_{0,P}$)

(1) Solve

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ (s_k)_1 & 0 & 0 & (x_k)_1 & 0 \\ 0 & (s_k)_2 & 0 & 0 & (x_k)_2 \end{bmatrix}}_{F'(x_k, \pi_k, s_k)} \begin{bmatrix} (\Delta x_k)_1 \\ (\Delta x_k)_2 \\ (\Delta \pi_k) \\ (\Delta s_k)_1 \\ (\Delta s_k)_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -(x_k)_1(s_k)_1 + \sigma \\ -(x_k)_2(s_k)_2 + \sigma \end{bmatrix}}_{-F(x_k, \pi_k, s_k)}$$

where $\mu_k = \frac{1}{2}\left((s_k)_1 (x_k)_1 + (s_k)_2 (x_k)_2\right)$

(2)

$$\begin{bmatrix} (x_{k+1})_1 \\ (x_{k+1})_2 \\ \pi_{k+1} \\ (s_{k+1})_1 \\ (s_{k+1})_2 \end{bmatrix} = \begin{bmatrix} (x_k)_1 \\ (x_k)_2 \\ \pi_k \\ (s_k)_1 \\ (s_k)_2 \end{bmatrix} + \alpha_k \begin{bmatrix} (\Delta x_k)_1 \\ (\Delta x_k)_2 \\ \Delta \pi_k \\ (\Delta s_k)_1 \\ (\Delta s_k)_2 \end{bmatrix}$$

where $\alpha_k > 0$ is s.t. $x_{k+1}, s_{k+1} \geq 0$.

PATH-FOLLOWING METHODS

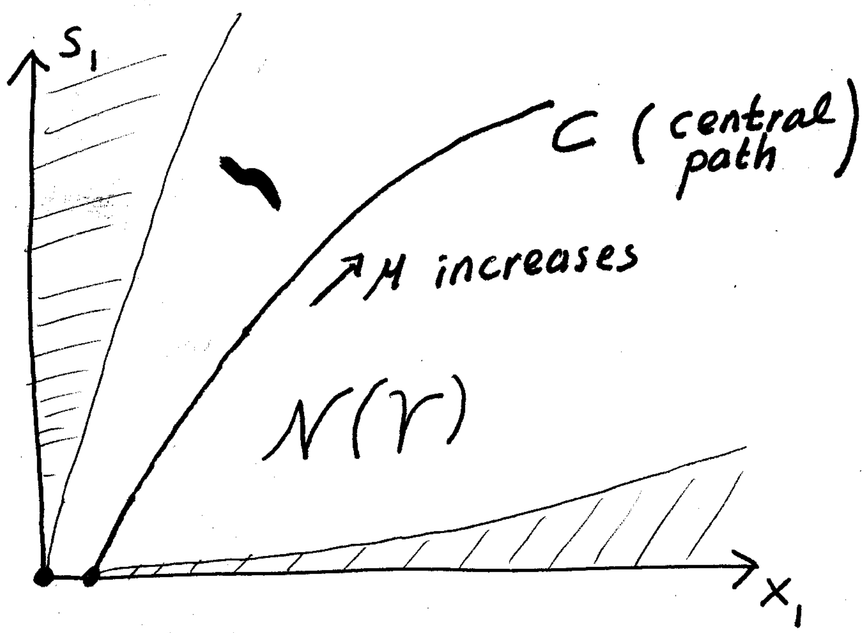
These are primal-dual methods forcing iterates to remain in a certain neighborhood of the central path.

For instance define

$$\mathcal{N}(\gamma) = \{(x, \pi, s) \in \mathbb{F}_{P,D} : x_j s_j \geq \mu \gamma, \; j = 1, \dots, n\}$$

where $\gamma \in (0,1)$ is a parameter.

γ is typically chosen close to 0
e.g. $\gamma = 10^{-3}$.



Given (x_k, π_k, s_k) we enforce $(x_{k+1}, \pi_{k+1}, s_{k+1}) \in \mathcal{N}(\gamma)$ within line search in Newton's method.

<u>ALGORITHM</u> (Long Path-Following)

Given $(x_0, \pi_0, s_0) \in \mathcal{N}(\gamma)$, $\gamma \in$ [redacted] $(0,1)$ and $\sigma_{min}, \sigma_{max} \in (0,1)$ s.t. $\sigma_{min} < \sigma_{max}$.

$k = 0$

While $\mu_k > \varepsilon$ (positive tolerance)

(1) Let $(\Delta x_k, \Delta \pi_k, \Delta s_k)$ be the solution of

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S_k & 0 & X_k \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta \pi_k \\ \Delta s_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -X_k S_k e + \sigma_k \mu_k \end{bmatrix}$$

where σ_k is chosen in $[\sigma_{min}, \sigma_{max}]$.

(2) Choose $\alpha_k \in (0, 1]$ the largest possible so that

$$\left(\begin{bmatrix} x_{k+1} \\ \pi_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ \pi_k \\ s_k \end{bmatrix} + \alpha_k \begin{bmatrix} \Delta x_k \\ \Delta \pi_k \\ \Delta s_k \end{bmatrix} \right) \in \mathcal{N}(\gamma)$$

(3) $k = k+1$

end

Convergence

The long path-following algorithm above is globally convergent assuming

* $(x_0, \pi_0, s_0) \in \mathcal{N}(\gamma)$
* $\gamma \in (0, 1)$
* σ_k are chosen in $[\sigma_{min}, \sigma_{max}]$

where $\sigma_{min}, \sigma_{max} \in (0,1)$ and $\sigma_{min} < \sigma_{max}$.

THM

Long path-following algorithm generates a sequence $(x_k, \pi_k, s_k) \in \mathcal{F}^{0}_{P,D}$ such that for some δ (independent of n and k)

$$\mu_{k+1} \leq \left(1 - \frac{\delta}{n}\right) \mu_k$$

holds for $k = 0, 1, \ldots$

Consequently $\lim_{k\to\infty}(x_k, \pi, s_k) \in \mathcal{F}_{P,D}$ and $\lim_{k\to\infty} \mu_k = 0$.

COROLLARY

Long path following algorithm generates a sequence $\{(x_k, \pi_k, s_k)\}$

* that satisfies KKT conditions in the limit as $k \to \infty$.

* Consequently

  (i) $\lim_{k\to\infty} x_k$ is a global minimizer for (LP).

  (ii) $\lim_{k\to\infty} (\pi_k, s_k) = (\pi_*, s_*)$ is a global maximizer for (DLP).