LECTURE 5

EIGENVALUE CHARACTERIZATION FOR POSITIVE DEFINITENESS

The condition

$* \; p^T A p > 0$ for all nonzero $p \in \mathbb{R}^{n \times n}$

can be characterized in terms of eigenvalues of A.

DEFN (Eigenvalue & Eigenvector)
A scalar $\lambda \in \mathbb{C}$ is called an eigenvalue of $A \in \mathbb{R}^{n\times n}$ if the equation

$$Ax = \lambda x$$

holds for some nonzero $x \in \mathbb{C}^n$. Such a vector $x \in \mathbb{C}^n$ is called an eigenvector associated with λ.

Eigenvalues and Characteristic Polynomial

$$Ax = \lambda x \quad \text{for some } x \neq 0$$

$$\Longleftrightarrow$$

$$(A - \lambda I)x = 0 \quad \text{for some } x \neq 0$$

$$\Longleftrightarrow \text{ (cols of } A-\lambda I \text{ are lin. depen.)}$$

$$(A - \lambda I) \text{ is not invertible}$$

$$\underbrace{p(\lambda) = \det(A - \lambda I)}_{\text{CHARACTERISTIC POLYNOMIAL OF } A} = 0$$

REMARK

* Eigenvalues of A are roots of its characteristic polynomial $p(\lambda) = \det(A - \lambda I)$

EXAMPLE

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)^2 - 16 \\ &= \lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3) \end{aligned}$$

Eigenvalues of A

$\lambda_1 = 5, \quad \lambda_2 = -3$

REMARK

* $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n with n roots (counting multiplicities).
* Consequently A has exactly n ~~roots~~ eigenvalues counting multiplicities.

EIGENVALUES OF SYMMETRIC MATRICES

$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$ has two real eigenvalues i.e. $\lambda_1 = 5$, $\lambda_2 = -3$

eigenvector v_1 associated with $\lambda_1 = 5$

$$(A - \lambda_1 I) v_1 = 0$$

$$\Longrightarrow \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} v_1 = 0 \Longrightarrow v_1 = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for any scalar $c \neq$

- eigenvector v_2 associated with $\lambda_2 = -3$

$$(A - \lambda_2 I) v_2 = 0$$

$$\Longrightarrow \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} v_2 = 0 \Longrightarrow v_2 = c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any scalar $c \neq 0$

Note that $v_1 \perp v_2$ (i.e. $v_1^T v_2 = 0$)

<u>THM</u> (Symmetric Eigenvalue Problem)
Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix (i.e. $A^T = A$) with eigenvalues $\lambda_1, \dots, \lambda_n$.

(i) $\lambda_1, \dots, \lambda_n$ are real.

(ii) The set $\{v_1, \dots, v_n\}$ of eigenvectors can be chosen orthogonal where v_i is an eigenvector associated with λ_i.

THM (Eigenvalues & Positive Definiteness)

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.

(i) $A \succ 0 \iff$ all eigenvalues of A are positive.

(ii) $A \succeq 0 \iff$ all eigenvalues of A are nonnegative.

PROOF OF (i)

$$\boxed{\text{Not all eigenvalues of } A \text{ are positive} \implies A \not\succ 0}$$

Suppose A has an eigenvalue $\lambda \leqslant 0$.
Let v be the eigenvector associated with λ.
Then

$$\begin{aligned} v^T A v &= v^T \lambda v \\ &= \lambda \|v\|_2^2 \leqslant 0. \end{aligned}$$

Consequently $A \not\succ 0$.

$$\boxed{\text{all eigenvalues of } A \text{ are positive} \implies A \succ 0}$$

Suppose all eigenvalues of A

$$\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$$

are positive. Denote the associated eigenvectors that are mutually orthogonal by

$$v_1, v_2, \ldots, v_n \in \mathbb{R}^n.$$

Any nonzero $p \in \mathbb{R}^n$ can be written of the form (since $\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$)

$$p = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ not all zero.

Consider

$$p^T A p = (\alpha_1 v_1^T + \alpha_2 v_2^T + \ldots + \alpha_n v_n^T) A (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n)$$

$$= (\alpha_1 v_1^T + \ldots + \alpha_n v_n^T)(\alpha_1 \underbrace{A v_1}_{\lambda_1 v_1} + \alpha_2 \underbrace{A v_2}_{\lambda_2 v_2} + \ldots + \alpha_n \underbrace{A v_n}_{\lambda_n v_n})$$

Since $v_i \perp v_j$ (that is $v_i^T v_j$) when $i \neq j$, we have

$$p^T A p = \alpha_1^2 \underbrace{\lambda_1}_{>0} \underbrace{(v_1^T v_1)}_{>0} + \alpha_2^2 \underbrace{\lambda_2}_{>0} \underbrace{(v_2^T v_2)}_{>0} + \ldots + \alpha_n^2 \underbrace{\lambda_n}_{>0} \underbrace{(v_n^T v_n)}_{>0}$$

> 0 (Since at least one of $\alpha_1, \ldots, \alpha_n$ is non-zero)

$\square$

EXAMPLE ①

$$f(x) = 3x_1^2 + 4x_1x_2 + 3x_2^2 + 8x_1 + 2x_2 + 9$$

$$= \frac{1}{2}x^T \underbrace{\begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}}_{A} x + \underbrace{[8 \ \ 2]}_{b^T} x + \underbrace{9}_{c}$$

Gradient

$$\nabla f(x) = \underbrace{\begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 8 \\ 2 \end{bmatrix}}_{b}$$

Hessian

$$\nabla^2 f(x) = \underbrace{\begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}}_{A}$$

(1) Stationary point

x_* s.t. $\nabla f(x_*) = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix} x_* + \begin{bmatrix} 8 \\ 2 \end{bmatrix} = 0$

$$\Longrightarrow$$

$$x_* = \begin{bmatrix} -2 \\ +1 \end{bmatrix}$$

(2) Positive Definiteness

Eigvalues of $\nabla^2 f(x_*) = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}$ are $\lambda_1 = 10$, $\lambda_2 = 2$

$$\Longrightarrow$$

$$\nabla^2 f(x_*) \succ 0$$

. By the second order sufficient conditions $x_* = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a local minimizer.

EXAMPLE 2

$$g(x) = 3x_1^2 + 4x_1x_2 - 3x_2^2 + 8x_1 + 14x_2 + 6$$
$$= \frac{1}{2} x^T \begin{bmatrix} 6 & 4 \\ 4 & -6 \end{bmatrix} x + \begin{bmatrix} 8 & 14 \end{bmatrix} x + 6$$

Gradient

$$\nabla g(x) = \begin{bmatrix} 6 & 4 \\ 4 & -6 \end{bmatrix} x + \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

Hessian

$$\nabla^2 g(x) = \begin{bmatrix} 6 & 4 \\ 4 & -6 \end{bmatrix}$$

(1) Stationary point

x_* s.t. $\nabla g(x_*) = \begin{bmatrix} 6 & 4 \\ 4 & -6 \end{bmatrix} x_* + \begin{bmatrix} 8 \\ 14 \end{bmatrix} = 0$

$$\Longrightarrow \quad x_* = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(2) Positive Semidefiniteness

Eigvalues of $\nabla^2 g(x_*) = \begin{bmatrix} 6 & 4 \\ 4 & -6 \end{bmatrix}$

are $\lambda_{1,2} = \pm 2\sqrt{13}$

$$\Longrightarrow$$

$$\nabla^2 g(x_*) \not\succeq 0$$

• By the second order necessary conditions $x_* = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is not a local minimizer.

Indeed g has no local minimizer.