MATH 171B — SPRING 2009
Solutions to Midterm 1

Q1

(a) Newton update rule

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{\cancel{x_k^p}\ (x_k)}{p\,\cancel{x_k^{p-1}}}$$

$$= \frac{p-1}{p} x_k$$

(b) Since $f'(x) = px^{p-1}$ and $f'(0) = 0$, we expect only linear convergence.

Therefore consider ($x_* = 0$ is the root)

$$\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = \lim_{k\to\infty} \frac{|(p-1)/p\ \cancel{x_k}|}{|\cancel{x_k}|}$$

$$= \frac{p-1}{p}$$

Therefore the rate of convergence is linear.

Q2

(a) Jacobian of F

$$F'(x) = \begin{bmatrix} 2x_1 & 2x_2 \\ 2(x_1-1) & 2(x_2-1) \end{bmatrix}$$

$$F'(1/2, 3/2) = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$$

Linear model about $(1/2, 3/2)$

$$L(x) = F'(1/2, 3/2)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}\right) + F(1/2, 3/2)$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}\left(\begin{bmatrix} x_1 - 1/2 \\ x_2 - 3/2 \end{bmatrix}\right) + \begin{bmatrix} 6/4 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1 - 1/2) + (3x_2 - 9/2) + 6/4 \\ (-x_1 + 1/2) + (x_2 - 3/2) - 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 3x_2 - 7/2 \\ -x_1 + x_2 - 3/2 \end{bmatrix} //$$

(b) A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called Lipschitz continuously differentiable if

$$\|F'(x) - F'(y)\| \leq \gamma \|x - y\| \text{ for all } x, y \in \mathbb{R}^n$$

and for some $\gamma \geq 0$.

To show that the specific function F is Lipschitz continuously differentiable

$$\|F'(x) - F'(y)\|_2 = 2 \left\| \begin{bmatrix} (x_1 - y_1) & (x_2 - y_2) \\ (x_1 - y_1) & (x_2 - y_2) \end{bmatrix} \right\|_2$$

(By triangle inequality)

$$\leq 2 \left\| \begin{bmatrix} x_1 - y_1 & 0 \\ x_1 - y_1 & 0 \end{bmatrix} \right\|_2 + 2 \left\| \begin{bmatrix} 0 & x_2 - y_2 \\ 0 & x_2 - y_2 \end{bmatrix} \right\|_2$$

$$= 2|x_1 - y_1| \left\| \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\|_2 + 2|x_2 - y_2| \left\| \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\|_2$$

$$\leq 2\|x - y\|_2 \sqrt{2} + 2\|x - y\|_2 \sqrt{2}$$

$$= 4\sqrt{2}\, \|x - y\|_2$$

(c) Jacobian at $x_* = (0, 1)$

$$F'(x_*) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

is not singular (since the columns of $F'(x_*)$ are linearly independent.)

Also from part (b) $F(x)$ is
[redacted] continuously differentiable.
Lipschitz

Newton's method converges q-quadratically when ① $F'(x_*)$ is not singular and ② $F(x)$ is Lipschitz continuously differentiable.

Therefore the sequence $\{x_k\}$ converges to $x_* = (0, 1)$ q-quadratically.

Q3

$$\nabla f(x) = \begin{bmatrix} 2x_1 + 4x_2 + 2 \\ 4x_1 + 8x_2 - 2 \end{bmatrix}$$

But the linear system

$$\begin{bmatrix} \overbrace{2x_1 + 4x_2 + 2}^{l_1(x)} \\ \underbrace{4x_1 + 8x_2 - 2}_{l_2(x)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is inconsistent, since $l_1(x)$, $l_2(x)$ represent two lines parallel to each other.

Since there does not exist a x_* such that $\nabla f(x_*) = 0$, <u>f has no local minimizer.</u>

$$\nabla g(x) = \begin{bmatrix} 2x_1 + 4 \\ 2x_2 - 2 \end{bmatrix}$$

$$\nabla g(x_*) = 0 \Longrightarrow \begin{bmatrix} 2x_1 + 4 \\ 2x_2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x_*} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

On the other hand

$$\nabla^2 g(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \underbrace{\begin{bmatrix} p_1 & p_2 \end{bmatrix}}_{p^T} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}}_{p}$$

$$= 2p_1^2 + 2p_2^2 > 0$$

for all $p \neq 0$.

$x_* = (-2, 1)$ is a local minimizer of g, since (i) $\nabla g(x_*) = 0$
(ii) $p^T \nabla^2 g(x_*) p > 0$ for all