MATH 409/509

SPRING 2011

SOLUTIONS TO MIDTERM

Q1. (a) $\{x_k\}$ is such that

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{(x_k^2 - 1/5)}{2x_k} = \frac{x_k^2 + 1/5}{2x_k}$$

(b) Order of convergence

$$\lim_{k\to\infty} \frac{\| x_{k+1} - 1/\sqrt{5} \|}{\| x_k - 1/\sqrt{5} \|^2} = \lim_{k\to\infty} \frac{\left| \frac{x_k^2 + 1/5}{2x_k} - \frac{1}{\sqrt{5}} \right|}{|x_k - 1/\sqrt{5}|^2}$$

$$= \lim_{k\to\infty} \frac{|x_k^2 - 2/\sqrt{5}\, x_k + 1/5|}{|2x_k|\,|x_k - 1/\sqrt{5}|^2}$$

$$= \lim_{k\to\infty} \frac{1}{|2x_k|} = \frac{\sqrt{5}}{2}$$

Therefore the order of convergence is quadratic.

(c)

1

$$\frac{d}{d\alpha} |f(x_k + \alpha p_k)| \Big|_{\alpha=0} = \frac{d}{d\alpha} \left(\sqrt{f^2(x_k + \alpha p_k)} \right) \Big|_{\alpha=0}$$

$$= \left(\frac{1}{2\sqrt{f^2(x_k + \alpha p_k)}} \, 2 f(x_k + \alpha p_k) f'(x_k + \alpha p_k) p_k \right) \Big|_{\alpha=0}$$

$$\left(\text{since } p_k = -\frac{f(x_k)}{f'(x_k)}\right) = \frac{f(x_k)}{|f(x_k)|} f'(x_k) \left(\frac{-f(x_k)}{f'(x_k)} \right) = -\frac{f^2(x_k)}{|f(x_k)|} < 0$$

Q2. (a)

$$\nabla f(x_*) = 0 \quad \text{and} \quad \nabla^2 f(x_*) \succ 0$$

$$\Longrightarrow$$

x_* is a local minimizer

(b)

$$f(x) = \frac{1}{2} x^T \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} x + x^T \begin{bmatrix} 7 \\ -3 \end{bmatrix} + 4$$

where $x = [x_1 \ \ x_2]^T$.

$$\nabla f(x) = \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

Stationary point x_* satisfies:

$$\nabla f(x_*) = 0 \Longleftrightarrow \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} x_* = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$$

$$\Longleftrightarrow x_* = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Hessian is positive definite, since it has positive eigenvalues.

$$\det\left(\begin{bmatrix} 6-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}\right) = 0 \Longleftrightarrow (6-\lambda)(2-\lambda) - 1 = 0$$

$$\Longleftrightarrow \lambda^2 - 8\lambda + 11 = 0$$

$$\Longleftrightarrow \lambda_{1,2} = \frac{8 \pm \sqrt{64-44}}{2} > 0$$

By second order sufficient conditions (since $\nabla f(x_*) = 0$ and $\nabla^2 f(x_*) \succ 0$)

$x_* = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a local minimizer.

(c) Let x_* be such that $\nabla f(x_*) = 0$.
Then by Taylor's thm

$$f(x) = f(x_*) + \underbrace{\nabla f(x_*)^T (x-x_*)}_{0} + \frac{1}{2} \underbrace{(x-x_*)^T \nabla_f^2(\tilde{x})^T (x-x_*)}_{q(x)}$$

Since $\underset{(=A)}{\nabla_f^2(\tilde{x})} \succeq 0$, $q(x) \geqslant 0$ for all x.

Consequently for all x

$$f(x) - f(x_*) = \frac{q(x)}{2} \geqslant 0$$

$$\Longrightarrow$$

$$f(x) \geqslant f(x_*)$$

meaning x_* is a global minimizer. □

Q3.

$$H_{k+1} = H_k + \sigma u u^T$$

$$\Longrightarrow$$

$$(*) \quad \underbrace{H_{k+1} y_k}_{s_k} = H_k y_k + \sigma u \underbrace{(u^T y_k)}_{\text{scalar}}$$

$$\Longrightarrow$$

$\boxed{u = \alpha (s_k - H_k y_k)}$ for some scalar α

Plug this again in $(*)$

$$s_k - H_k y_k = \sigma \alpha^2 (s_k - H_k y_k) \left((s_k - H_k y_k)^T y_k \right)$$

$$\Longrightarrow$$

$$\boxed{\sigma \alpha^2 = 1 / \left((s_k - H_k y_k)^T y_k \right)}$$

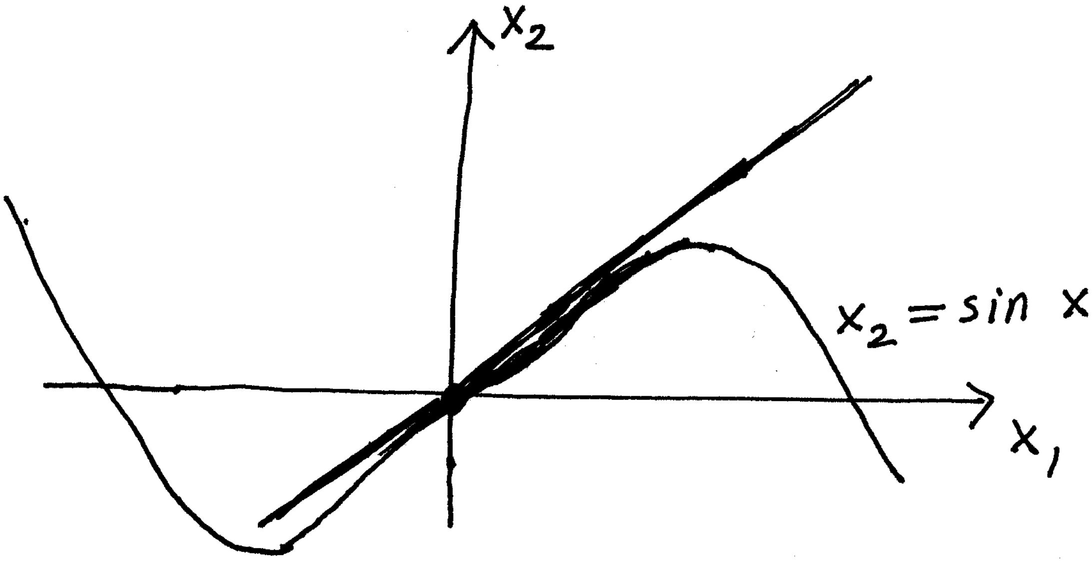
Consequently

$$H_{k+1} = H_k + \sigma\alpha^2 (s_k - H_k y_k)(s_k - H_k y_k)^T$$

$$= H_k + \frac{(s_k - H_k y_k)(s_k - H_k y_k)^T}{((s_k - H_k y_k)^T y_k)}$$

Q4. (a)

Tangent Cone at 0



Consists of the set of tangent vectors to feasible paths at 0

equivalently

all vectors tangent to $x_2 = \sin x_1$ at 0.

$$T^\circ(0) = \left\{ \alpha \begin{bmatrix} x_1 \\ \sin x_1 \end{bmatrix}' \Bigg|_{x_1=0} : \alpha \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{bmatrix} 1 \\ \cos x_1 \end{bmatrix} \Bigg|_{x_1=0} : \alpha \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Null space of the constraint Jacobian at 0

$$J(x) = [-\cos x_1 \quad 1]$$

$$\Rightarrow J(0) = [-1 \quad 1]$$

$$\Rightarrow \text{Null}(J(0)) = \{p \in \mathbb{R}^2 : J(0)p = 0\}$$

$$= \left\{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R}\right\}$$

$$= \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$T^{\circ}(0) = \text{Null}(J(0))$$

CONSTRAINT QUALIFICATION HOLDS

(b) $\boxed{\text{First order optimality condition}}$

~~For~~ If $x \in \mathbb{R}^2$ is a local minimizer, then

(i) $\nabla f(x) = J(x)^T \lambda$ for some λ $\Rightarrow \begin{bmatrix} 6x_1 \\ 2(x_2-1) \end{bmatrix} = \begin{bmatrix} -\cos x_1 \\ 1 \end{bmatrix}$ for some λ

(ii) $c(x) = x_2 - \sin(x_1) = 0$

At $\bar{x} = 0$

(ii) $c(\bar{x}) = 0 - \sin(0) = 0$

but

$$\text{(i)} \quad \nabla f(\bar{x}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \neq \overbrace{\begin{bmatrix} -\cos 0 \\ 1 \end{bmatrix}}^{J(\bar{x})^T} \lambda$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \lambda \text{ for all}$$

Consequently $\bar{x} = (0,0)$ is not a local minimizer. (by first order optimality condition