MATH 171B – SPRING 2009

Solutions to Midterm 2

Q1) Newton direction p^N satisfies

$$\nabla^2 Q(x_0) p^N = -\nabla Q(x_0)$$

where

$$\nabla Q(x_0) = 2x_0,$$
$$\nabla^2 Q(x_0) = 2I.$$

It follows that

$$2p^N = -2x_0$$
$$\Longrightarrow p^N = -x_0$$

implying

$$x_1 = x_0 + p^N = 0,$$

which is the unique local minimizer of Q.

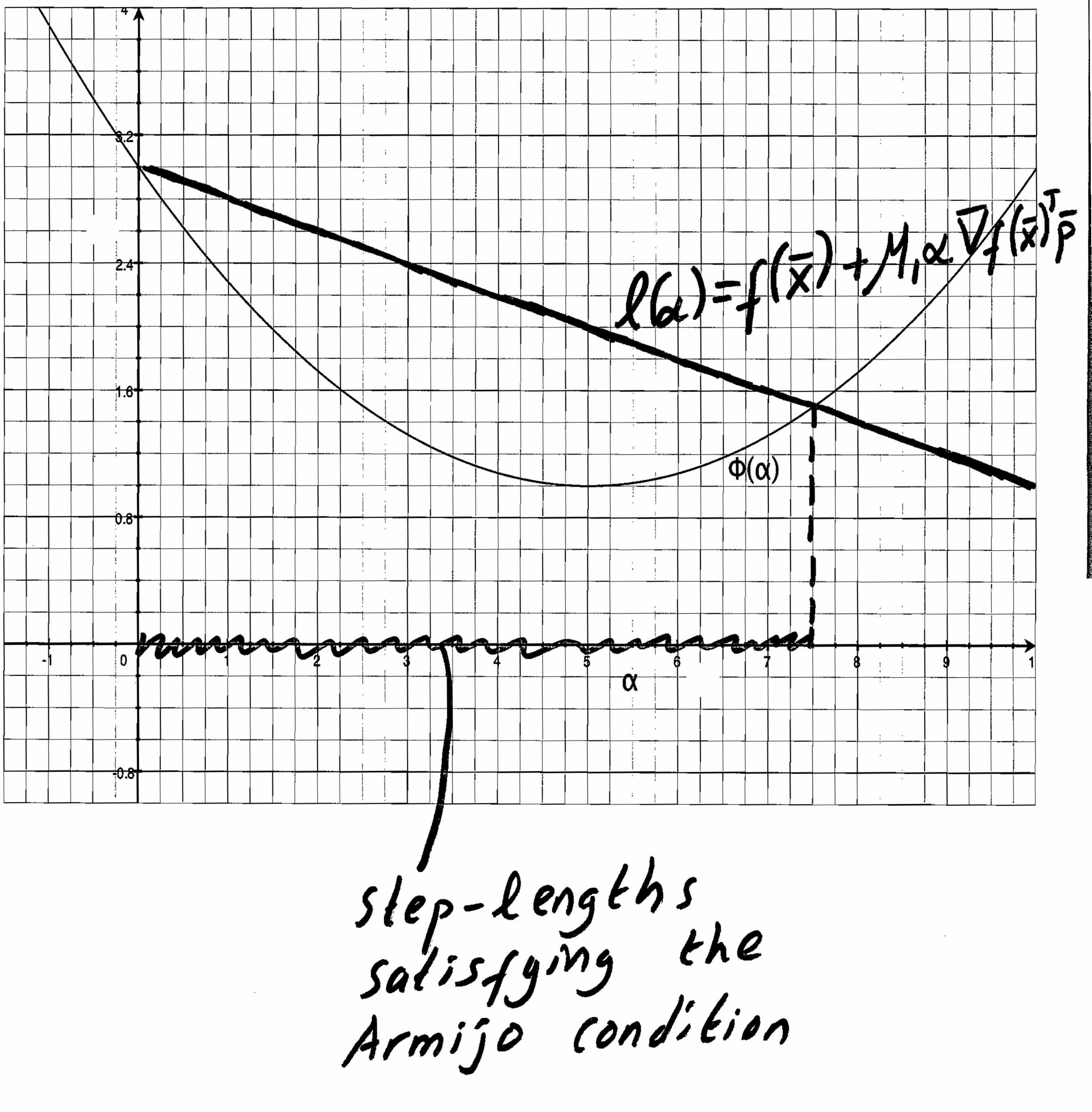
Q2)

(a) Armijo sufficient decrease condition.

$$\underbrace{f(\bar{x} + \alpha \bar{p})}_{\phi(\alpha)} \leqslant \underbrace{f(\bar{x}) + \mu_1 \alpha \nabla f(\bar{x})^T \bar{p}}_{l(\alpha)}$$ (where $\mu_1 \in (0,1)$)

(b) Since $\phi'(0) = -0.8$ and the slope of $l(\alpha) = \mu_1 \phi'(0)$, the slope of $l(\alpha)$ is -0.2 (SEE THE PLOT)

Graph for question 2(b)



(1b)

Q3) (a) Eigvalues of A

All λ s.t. $0 = \det(A-\lambda I) = \det\left(\begin{bmatrix} -1-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}\right)$

$$= (\lambda+1)^2 - 3$$
$$= \lambda^2 + 2\lambda - 8$$
$$= (\lambda+4)(\lambda-2)$$

$\lambda_1 = -4$, $\lambda_2 = 2$ are the eigvalues of A.

eigvec v_1 assoc with $\lambda_1 = -4$

$$0 = \begin{bmatrix} -1-(-4) & 3 \\ 3 & -1-(-4) \end{bmatrix} v_1 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} v_1$$

$$\Longrightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ or } v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \text{(Normalize)}$$

eigvec v_2 assoc with $\lambda_2 = 2$

$$0 = \begin{bmatrix} -1-(2) & 3 \\ 3 & -1-(2) \end{bmatrix} v_2 = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} v_2$$

$$\Longrightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ or } v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Orthogonal eigvalue decomposition

$$A = \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}}_{V^T}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(b) $$\hat{A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

(c) $$\hat{A}\, p_m = -\nabla f(\bar{x})$$

$$\Longrightarrow \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} p_m = \begin{bmatrix} +4 \\ -4 \end{bmatrix} \Longrightarrow p_m = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

p_m is a descent direction, since

$$\nabla f(\bar{x})^T p_m = \begin{bmatrix} -4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -8 < 0$$

3

Q4) By Taylor's theorem with integral remainder

$$\nabla f(x_{k+1}) = \nabla f(x_k) + \int_0^1 \nabla^2 f(x_k + t s_k) s_k \, dt$$

where $s_k = x_{k+1} - x_k$.

Multiplying both sides with s_k^T from the left and rearranging yield

$$\underbrace{s_k^T \left(\nabla f(x_{k+1}) - \nabla f(x_k)\right)}_{y_k} = \int_0^1 \underbrace{s_k^T \nabla^2 f(x_k + t s_k) s_k}_{\substack{>0 \text{ for all } t \\ \left(\text{since } \nabla^2 f(x) \succ 0 \text{ at all } x\right)}} dt$$

$$> 0.$$

$\square$