Basic Linear Algebra Background

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Definition 0.1 (Vector Space)**.** *A vector space V is a set (over a field $\mathbb{F}$) that comes with an addition $(+)$ and a multiplication with scalars $(\cdot)$ such that*

(1) $v + w \in V$ *for all* $v, w \in V$,

(2) $\alpha \cdot v \in V$ *for all* $v \in V$ *and for all* $\alpha \in \mathbb{F}$.

The addition must satisfy the following properties:

(A1) $v + w = w + v$ *for all* $v, w \in V$.

(A2) $u + (v + w) = (u + v) + w$ *for all* $u, v, w \in V$.

(A3) *There exists a* $0 \in V$ *such that* $v + 0 = v$ *for all* $v \in V$.

(A4) *For every* $v \in V$ *there exists* $-v \in V$ *such that* $v + (-v) = 0$.

The multiplication with scalars must satisfy the following:

(M1) $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$ *for all* $\alpha, \beta \in \mathbb{F}$ *and for all* $v \in V$.

(M2) $\alpha \cdot (v + w) = \alpha \cdot v + \alpha \cdot w$ *for all* $\alpha \in \mathbb{F}$ *and for all* $v, w \in V$.

(M3) $\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v$ *for all* $\alpha, \beta \in \mathbb{F}$ *and for all* $v \in V$.

(M4) *There exists* $1 \in \mathbb{F}$ *such that* $1 \cdot v = v$ *for all* $v \in V$.

Example.
The subset
$$\mathcal{P} := \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
is a vector space over $\mathbb{R}$.

In particular, for every $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathcal{P}$, we have
$$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = \underbrace{(x_1 + y_1 + z_1)}_{0} + \underbrace{(x_2 + y_2 + z_2)}_{0} = 0,$$
so $(x_1, y_1, z_1) + (x_2, y_2, z_2) = ((x_1 + x_2), (y_1 + y_2), (z_1 + z_2)) \in \mathcal{P}$.

Additionally, for every $\alpha \in \mathbb{R}$ and for every $(x, y, z) \in \mathcal{P}$, we have
$$\alpha x + \alpha y + \alpha z = \alpha \underbrace{(x + y + z)}_{0} = 0,$$
so $\alpha \cdot (x, y, z) = (\alpha x, \alpha y, \alpha z) \in \mathcal{P}$.

Definition 0.2 (Subspace)**.** *A subspace $\mathcal{S}$ of a vector space $\mathcal{V}$ is a subset of $\mathcal{V}$ that is also a vector space.*

Example. $\mathcal{P} := \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ is a subspace of $\mathbb{R}^3$.

Definition 0.3 (Span)**.** *The span of a set of vectors* $\{v_1, \ldots, v_n\}$ *in a vector space* V *(over* $\mathbb{F}$*) is defined by*

$$\operatorname{span}\{v_1, \ldots, v_n\} \;:=\; \{\alpha_1 \cdot v_1 + \cdots + \alpha_n \cdot v_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{F}\}.$$

$$\begin{aligned} \mathcal{P} \;&:=\; \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\} \\ &=\; \{(x, y, -y - x) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\} \\ &=\; \{x \cdot (1, 0, -1) + y \cdot (0, 1, -1) \mid x, y \in \mathbb{R}\} \\ &=\; \operatorname{span}\{(1, 0, -1), (0, 1, -1)\} \end{aligned}$$

Definition 0.4 (Linear Independence)**.** *A set of vectors* $\{v_1, \ldots, v_n\}$ *in a vector space* V *(over* $\mathbb{F}$*) is linearly independent if*

$$\alpha_1 \cdot v_1 + \cdots + \alpha_n \cdot v_n \;=\; 0$$

holds only for $\alpha_1 = \cdots = \alpha_n = 0$*.*

The set $\{v_1, \ldots, v_n\}$ *is linearly dependent if*

$$\alpha_1 \cdot v_1 + \cdots + \alpha_n \cdot v_n \;=\; 0$$

holds for some $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ *not all zero.*

Example. The set $\{(1,0,-1),(0,1,-1)\}$ is linearly independent, because

$$\begin{aligned} 0 = \alpha_1 \cdot (1,0,-1) + \alpha_2 \cdot (0,1,-1) = (\alpha_1, \alpha_2, -\alpha_1 - \alpha_2) \\ \implies \quad \alpha_1 = \alpha_2 = 0. \end{aligned}$$

On the other hand, $\{\underbrace{(1,0,-1)}_{v_1}, \underbrace{(0,1,-1)}_{v_2}, \underbrace{(-5,3,2)}_{v_3}\}$ is linearly dependent,

$$-5 \cdot v_1 + 3 \cdot v_2 \;=\; v_3.$$

Definition 0.5 (Basis)**.** *A basis* B *for a vector space* V *is a set such that*

(1) $\operatorname{span} B = V$*, and*

(2) B *is linearly independent.*

- $\operatorname{span}\{(1,0,-1),(0,1,-1)\} = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$.
- $\{(1,0,-1),(0,1,-1)\}$ is linearly independent.

$\{(1,0,-1),(0,1,-1)\}$ is a basis for $\mathcal{P} := \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$.

Theorem 0.6. *Let B_1 and B_2 be two bases for a vector space V. Then*

$$\#B_1 \;=\; \#B_2.$$

Example. $\{(1,0,-1),(0,1,-1)\}$ and $\{(-1,0,1),(-1,1,0)\}$ are both bases for $\mathcal{P} := \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$.

Definition 0.7 (Dimension)**.** *The dimension of a vector space V is defined by*

$$dim\, V \;:=\; \#B,$$

where B is any basis for V.