Reminders from Lecture 9 & Orthogonal Projectors

Projections & Projectors

Given $\mathcal{S}_1, \mathcal{S}_2$, subspaces of $\mathbb{C}^n$ such that

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathbb{C}^n$$

- $v = v_{S_1} + v_{S_2}, \quad v_{S_1} \in \mathcal{S}_1,\ v_{S_2} \in \mathcal{S}_2.$
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  v_{S_1} is the projection of v onto $\mathcal{S}_1$ along $\mathcal{S}_2$.
- The matrix $P \in \mathbb{C}^{n \times n}$ such that

$$Pv = v_{S_1} \quad \forall v \in \mathbb{C}^n$$

  is the projector onto $\mathcal{S}_1$ along $\mathcal{S}_2$.

Characterization of Projectors

- If P is a projector, $P^2 = P$.
- If $P^2 = P$, then
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Example.

$$P = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

As $P^2 = P$, this is a projector.
P projects onto $\mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ along $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Orthogonal Projections & Orthogonal Projectors

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$$v = v_S + v_{S^\perp},$$

for some $v_S \in \mathcal{S}$ and $v_{S^\perp} \in \mathcal{S}^\perp$ in a *unique* way.
(since $\mathcal{S} \oplus \mathcal{S}^\perp = \mathbb{C}^n$)

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- The matrix $P \in \mathbb{C}^{n \times n}$ such that

$$Pv = v_S \quad \forall v \in \mathbb{C}^n$$

is the orthogonal projector onto $\mathcal{S}$.