Orthogonal Projectors

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$\mathcal{S}$ - a subspace of $\mathbb{C}^n$

Every vector $v \in \mathbb{C}^n$ can be expressed in a <u>unique</u> way as

$$v = v_S + v_{S^\perp}, \quad \exists v_S \in \mathcal{S},\ \exists v_{S^\perp} \in \mathcal{S}^\perp.$$

- v_S is the orthogonal projection of v onto $\mathcal{S}$.
- The matrix $P \in \mathbb{C}^{n \times n}$ such that

  $$Pv = v_S \quad \forall v \in \mathbb{C}^n$$

  is the orthogonal projector onto $\mathcal{S}$.

Representation in terms of a basis $\{a_1, \ldots, a_q\}$ for $\mathcal{S}$.

$$P = A(A^*A)^{-1}A^*, \qquad A := \begin{bmatrix} a_1 & a_2 & \ldots & a_q \end{bmatrix}$$

Example.

$$\mathcal{S} := \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}, \quad \text{Orthogonal projector onto } \mathcal{S}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \left(\frac{1}{8} \begin{bmatrix} 6 & -4 \\ -4 & 4 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Representation in terms of an orthonormal basis $\{u_1, \ldots, u_q\}$

$$P \;=\; UU^*, \qquad U := \begin{bmatrix} u_1 & u_2 & \ldots & u_q \end{bmatrix}$$

Example.

$$\mathcal{S} := \operatorname{span}\left\{ \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tfrac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \text{Orthogonal proj. onto } \mathcal{S}$$

$$P \;=\; \begin{bmatrix} 1/2 & 0 \\ 1/2 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \;=\; \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Theorem. Let $P \in \mathbb{C}^{n \times n}$ be a projector. The following are equivalent:

(1) P is an orthogonal projector.

(2) $P^* = P$.

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Suppose P is a projector onto the subspace $\mathcal{S}$ with the orthonormal basis $\{u_1, \ldots, u_q\}$. Then

$$P \quad = \quad UU^*, \quad \text{where} \quad U := \left[\begin{array}{ccc} u_1 & \ldots & u_q \end{array}\right]$$

satisfies $P^* = P$.

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Proof of (2) $\Longrightarrow$ (1)

P is a projector onto Col(P) along Null(P).
Hence, it suffices to show that Null(P) = Col$(P)^{\perp}$.

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Let us prove Null$(P) \subseteq$ Col$(P)^\perp$.
Let $z \in$ Null(P).

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(1) P is an orthogonal projector.

(2) $P^* = P$.

Proof of (2) $\Longrightarrow$ (1)

Let us prove $\text{Null}(P) \subseteq \text{Col}(P)^{\perp}$.
Let $z \in \text{Null}(P)$. For every $y \in \text{Col}(P)$

$$y^* z \;=\; (P\widetilde{y})^* z \;=\; \widetilde{y}^* P^* z \;=\; \widetilde{y}^* P z = 0$$

for some $\widetilde{y} \in \mathbb{C}^n$, so $z \in \text{Col}(P)^{\perp}$.

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(1) P is an orthogonal projector.

(2) $P^* = P$.

Proof of (2) $\implies$ (1)

Now let us prove $\text{Null}(P) \supseteq \text{Col}(P)^\perp$.
Let $z \in \text{Col}(P)^\perp$. Then

$$0 \;=\; (Pz)^* z \;=\; z^* P z \;=\; z^* P^2 z \;=\; \|Pz\|_2^2$$

implying $Pz = 0$, so $z \in \text{Null}(P)$.