QR Factorization by Householder Reflectors

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Full QR Factorization

Given $A \in \mathbb{C}^{n \times p}$ with $n \geq p$,

$$A = QR$$

- $Q \in \mathbb{C}^{n \times n}$ is unitary,
- $R \in \mathbb{C}^{n \times p}$ is upper triangular

Approach

$$Q_n \dots Q_2 Q_1 A \quad = \quad R$$

- $Q_j \in \mathbb{C}^{n \times n}$ - (unitary and Hermitian)
  Householder reflector to introduce 0 on the jth column of A below (j,j) entry.

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  Householder reflector to introduce 0 on the jth column of A below (j,j) entry.

This will yield

$$A = \underbrace{Q_1 Q_2 \dots Q_n}_{Q} R.$$

Householder reflector Q achieving

$$v \quad \mapsto \quad \begin{bmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \|v\|_2 e_1 = Qv$$

$$Q \quad = \quad I_n - 2qq^*, \qquad q \quad = \quad \frac{v - \|v\|_2 e_1}{\| \; v - \|v\|_2 e_1 \; \|_2}$$

kth step of the algorithm

$$A^{(k)} = \begin{bmatrix} x & & & x & \dots & x \\ & \ddots & & & & \\ & & x & x & & x \\ 0 & & 0 & {\color{red}x} & & {\color{blue}x} \\ 0 & & 0 & {\color{blue}x} & & {\color{blue}x} \\ & & \vdots & {\color{blue}\vdots} & & {\color{blue}\vdots} \\ 0 & & 0 & {\color{blue}x} & & {\color{blue}x} \end{bmatrix} \mapsto A^{(k+1)} = \underbrace{\begin{bmatrix} x & & & x & \dots & x \\ & \ddots & & & & \\ & & x & x & & x \\ 0 & & 0 & {\color{red}x} & & {\color{blue}x} \\ 0 & & 0 & {\color{blue}0} & & {\color{blue}x} \\ & & \vdots & {\color{blue}\vdots} & & {\color{blue}\vdots} \\ 0 & & 0 & {\color{blue}0} & & {\color{blue}x} \end{bmatrix}}_{Q_k A^{(k)}}$$

x - (k, k) entry

kth step of the algorithm

$$A^{(k)} = \begin{bmatrix} x & & & x & \dots & x \\ & \ddots & & & & \\ & & x & x & & x \\ 0 & & 0 & x & & x \\ 0 & & 0 & x & & x \\ & & \vdots & \vdots & & \vdots \\ 0 & & 0 & x & & x \end{bmatrix} \mapsto A^{(k+1)} = \underbrace{\begin{bmatrix} x & & & x & \dots & x \\ & \ddots & & & & \\ & & x & x & & x \\ 0 & & 0 & x & & x \\ 0 & & 0 & 0 & & x \\ & & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & & x \end{bmatrix}}_{Q_k A^{(k)}}$$

x - (k,k) entry

$$Q_k := \begin{bmatrix} I_{k-1} & 0 \\ 0 & I - 2q_k q_k^* \end{bmatrix}, \qquad q_k := \frac{A(k:n,k) - \|A(k:n,k)\|_2 e_1}{\|\, A(k:n,k) - \|A(k:n,k)\|_2 e_1 \,\|_2}$$

QR Factorization by Householder Reflectors

for $k = 1, \ldots, p$ **do**

  $h_k \leftarrow A(k:n, k)$

  $q_k \leftarrow \{h_k - \|h_k\|_2 e_1\} \,/\, \{\|h_k - \|h_k\|_2 e_1\|\}$

  $A(k:n, k:p) \leftarrow A(k:n, k:p) - 2q_k(q_k^* A(k:n, k:p))$

end for

$R \leftarrow A$

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