Matrix Norms

Matrix p-Norm, $p \geq 1$

Recall the matrix 2-norm for a given $A \in \mathbb{C}^{m \times n}$

$$\|A\|_2 \quad := \quad \max_{v \in \mathbb{C}^n, \|v\|_2 = 1} \|Av\|_2 \quad = \quad \sigma_1(A)$$

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Definition (Matrix p-norm)

Let $A \in \mathbb{C}^{m \times n}$.

$$\|A\|_p := \max_{v \in \mathbb{C}^n, \|v\|_p = 1} \|Av\|_p$$

(Also called the matrix-norm induced by the vector p-norm.)

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- Most widely used ones: 1-norm, 2-norm, ∞-norm

Matrix 1-Norm

Theorem (Characterization of the 1-norm)

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$$\|A\|_1 = \max_{j=1,\ldots,n} \|a_j\|_1 .$$

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For every $A \in \mathbb{C}^{m\times n}$, we have

$$\|A\|_1 = \max_{j=1,\dots,n} \|a_j\|_1 .$$

Ex.

$$\left\| \begin{bmatrix} -3 & 2 & 7 \\ 2 & -9 & 0 \\ 1 & 3 & 5 \end{bmatrix} \right\|_1 = 14$$

Matrix 1-Norm

Proof.
Let $k \in \{1, \ldots, n\}$ be such that $\|a_k\|_1 = \max_{j=1,\ldots,n} \|a_j\|_1$.

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Consider any $v \in \mathbb{C}^n$ with $\|v\|_1 = 1$.

$$
\begin{aligned}
\|Av\|_1 &= \|v_1 a_1 + v_2 a_2 + \cdots + v_n a_n\|_1 \\
&\leq |v_1|\|a_1\|_1 + |v_2|\|a_2\|_1 + \cdots + |v_n|\|a_n\|_1 \\
&\leq \underbrace{(|v_1| + |v_2| + \cdots + |v_n|)}_{\|v\|_1 = 1} \|a_k\|_1 \quad = \quad \|a_k\|_1 .
\end{aligned}
$$

Matrix 1-Norm

Proof.
Let $k \in \{1, \ldots, n\}$ be such that $\|a_k\|_1 = \max_{j=1,\ldots,n} \|a_j\|_1$.

Consider any $v \in \mathbb{C}^n$ with $\|v\|_1 = 1$.

$$\begin{aligned} \|Av\|_1 &= \|v_1 a_1 + v_2 a_2 + \cdots + v_n a_n\|_1 \\ &\leq |v_1|\|a_1\|_1 + |v_2|\|a_2\|_1 + \cdots + |v_n|\|a_n\|_1 \\ &\leq \underbrace{(|v_1| + |v_2| + \cdots + |v_n|)}_{\|v\|_1 = 1} \|a_k\|_1 = \|a_k\|_1. \end{aligned}$$

This shows that

$$\|A\|_1 \leq \|a_k\|_1 = \max_{j=1,\ldots,n} \|a_j\|_1.$$

Matrix 1-Norm

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This completes the proof of

$$\left(\|A\|_1 \;=\; \max_{v \in \mathbb{C}^n, \|v\|_2 = 1} \|Av\|_1\right) \;=\; \left(\|a_k\|_1 = \max_{j=1,\ldots,n} \|a_j\|_1\right).$$

Matrix ∞-Norm

Theorem (Characterization of the ∞-norm)

For every $A \in \mathbb{C}^{m \times n}$, we have

$$\|A\|_\infty = \max_{j=1,\ldots,m} \|A(j,:)^T\|_1 .$$

Matrix ∞-Norm

Theorem (Characterization of the ∞-norm)

For every $A \in \mathbb{C}^{m\times n}$, we have

$$\|A\|_\infty = \max_{j=1,\dots,m} \|A(j,:)^T\|_1.$$

Ex.

$$\left\| \begin{bmatrix} -3 & 2 & 7 \\ 2 & -9 & 0 \\ 1 & 3 & 5 \end{bmatrix} \right\|_\infty = 12$$

Frobenius Norm

Given $A \in \mathbb{C}^{m \times n}$,

$$\|A\|_F \; := \; \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk}^2} \; = \; \sqrt{\text{trace}(A^T A)}.$$

- Not induced by any vector norm.