Power Iteration

Sequence by the Power Iteration

Given $A \in \mathbb{C}^{n \times n}$ and $q^{(0)} \in \mathbb{C}^n$,

Power iteration generates a sequence $\{q^{(k)}\}$ such that

$$q^{(k+1)} \quad := \quad \frac{Aq^{(k)}}{\|Aq^{(k)}\|_2}.$$

The Sequence as $k \to \infty$

Assumption: A has n linearly independent eigenvectors.

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Notations

- $\lambda_1, \ldots, \lambda_n$ denotes eigenvalues such that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

- $v^{(j)}$ is the eigenvector corresponding to λ_j.

The Sequence as $k \to \infty$

Notice

$$q^{(k)} = \frac{Aq^{(k-1)}}{\|Aq^{(k-1)}\|_2} = \frac{A^k q^{(0)}}{\|A^k q^{(0)}\|_2}$$

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$$q^{(0)} = \alpha_1 v^{(1)} + \alpha_2 v^{(2)} + \cdots + \alpha_n v^{(n)}$$

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$$\begin{aligned} q^{(0)} &= \alpha_1 v^{(1)} + \alpha_2 v^{(2)} + \cdots + \alpha_n v^{(n)} \\ Aq^{(0)} &= \alpha_1 \lambda_1 v^{(1)} + \alpha_2 \lambda_2 v^{(2)} + \cdots + \alpha_n \lambda_n v^{(n)} \end{aligned}$$

The Sequence as $k \to \infty$

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Aq^{(0)} &= \alpha_1 \lambda_1 v^{(1)} + \alpha_2 \lambda_2 v^{(2)} + \cdots + \alpha_n \lambda_n v^{(n)} \\
A^2 q^{(0)} &= \alpha_1 \lambda_1^2 v^{(1)} + \alpha_2 \lambda_2^2 v^{(2)} + \cdots + \alpha_n \lambda_n^2 v^{(n)}
\end{aligned}$$

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A^2 q^{(0)} &= \alpha_1 \lambda_1^2 v^{(1)} + \alpha_2 \lambda_2^2 v^{(2)} + \cdots + \alpha_n \lambda_n^2 v^{(n)} \\
&\vdots \\
A^k q^{(0)} &= \alpha_1 \lambda_1^k v^{(1)} + \alpha_2 \lambda_2^k v^{(2)} + \cdots + \alpha_n \lambda_n^k v^{(n)}
\end{aligned}$$

The Sequence as $k \to \infty$

$$q^{(k)} = \frac{A^k q^{(0)}}{\|A^k q^{(0)}\|_2} = \alpha_1 \lambda_1^k s_k v^{(1)} + \alpha_2 \lambda_2^k s_k v^{(2)} + \cdots + \alpha_n \lambda_n^k s_k v^{(n)}$$

where $s_k := 1/\|A^k q^{(0)}\|_2$.

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where $s_k := 1/\|A^k q^{(0)}\|_2$.

Further assume

- $|\lambda_1| > |\lambda_2|$,
- $\alpha_1 \neq 0$.

Blue term dominates the right-hand side for large k.