Convergence of Power Iteration

Rate-of-Convergence

A sequence $\{q^{(k)}\}$ such that $\lim_{k\to\infty} q^{(k)} = q_*$

- Converges *linearly* if

$$\lim_{k\to\infty} \frac{\|q^{(k+1)} - q_*\|_2}{\|q^{(k)} - q_*\|_2} = \mu$$

for some $\mu \in (0, 1)$.
(e.g. $\{1/2^k\}$ converges to 0 linearly)

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- Converges *superlinearly* if

$$\lim_{k\to\infty} \frac{\|q^{(k+1)} - q_*\|_2}{\|q^{(k)} - q_*\|_2} = 0$$

(e.g. $\{1/k!\}$ converges to 0 superlinearly)

Rate-of-Convergence

- Converges *quadratically* if

$$\lim_{k\to\infty} \frac{\|q^{(k+1)} - q_*\|_2}{\|q^{(k)} - q_*\|_2^2} = \mu$$

for some $\mu > 0$.
(e.g. $\{1/10^{(2^k)}\}$ converges to 0 quadratically)

Rate-of-Convergence

- Converges *quadratically* if

$$\lim_{k\to\infty} \frac{\|q^{(k+1)} - q_*\|_2}{\|q^{(k)} - q_*\|_2^2} = \mu$$

for some $\mu > 0$.
(e.g. $\{1/10^{(2^k)}\}$ converges to 0 quadratically)

- Converges *with order $p > 1$* if

$$\lim_{k\to\infty} \frac{\|q^{(k+1)} - q_*\|_2}{\|q^{(k)} - q_*\|_2^p} = 0$$

for some $\mu > 0$.
(e.g. $\{1/10^{(3^k)}\}$ converges to 0 with order 3)

Convergence of Power Iteration

Given $A \in \mathbb{C}^{n \times n}$ and $q^{(0)} \in \mathbb{C}^n$,

Power iteration generates a sequence $\{q^{(k)}\}$ such that

$$q^{(k+1)} := \frac{Aq^{(k)}}{\|Aq^{(k)}\|_2}.$$

- $\lambda_1, \ldots, \lambda_n$ denotes eigenvalues such that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

- $v^{(j)}$ is the eigenvector corresponding to λ_j.

Convergence of Power Iteration

Convergence

There exists a sequence $\{c_k\}$ of real numbers such that $\lim_{k\to\infty} |c_k| = 1$ and

$$\|q^{(k)} - c_k v^{(1)}\|_2 \to 0 \quad \text{as} \quad k \to \infty.$$

We write

$$\text{span}\{q^{(k)}\} \longrightarrow \text{span}\{v^{(1)}\} \quad \text{as} \quad k \to \infty.$$

Convergence of Power Iteration

Rate-of-Convergence

$$\frac{\|q^{(k+1)} - c_{k+1} v^{(1)}\|_2}{\|q^{(k)} - c_k v^{(1)}\|_2} = \left|\frac{\lambda_2}{\lambda_1}\right|$$

Convergence of Power Iteration

Ex.

$$A = \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda_1 = -3$, $\lambda_2 = 2$, $\lambda_3 = 1$, $v_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$

Convergence of Power Iteration

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- Convergence

$$\text{span}\{q^{(k)}\} \longrightarrow \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

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- Convergence

$$\text{span}\{q^{(k)}\} \longrightarrow \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

- Rate-of-convergence

$$\frac{\|q^{(k+1)} - c_{k+1} v^{(1)}\|_2}{\|q^{(k)} - c_k v^{(1)}\|_2} = \frac{2}{3}$$