Why the QR Algorithm Works

A Generalization of the Power Iteration

Simultaneous Power Iteration

(1) $\widetilde{Q}_0$ - matrix with orthonormal columns

(2) $\widetilde{Q}_{k+1}\widetilde{R}_{k+1} = H\widetilde{Q}_k$ (a reduced QR factorization on the left)

(3) $H_k = \widetilde{Q}_k^* H \widetilde{Q}_k$

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- $Q_{k+1} = \widetilde{Q}_k^* \widetilde{Q}_{k+1} \implies \widetilde{Q}_{k+1} = \widetilde{Q}_k Q_{k+1}$

Equivalence to the Simultaneous Power Iteration

Recalling one iteration of the QR algorithm

$$H_k = Q_{k+1} R_{k+1}, \quad H_{k+1} = R_{k+1} Q_{k+1}$$

Theorem

Simultaneous power iteration with $\widetilde{Q}_0 = I_n$ is equivalent to the QR algorithm (without shifts). In particular,

(i) *both generate the same sequence $\{H_k\}$,*

(ii) $\widetilde{Q}_k \; = \; Q_1 Q_2 \dots Q_k$.