Gram-Schmidt and Eigenvalues (L3)

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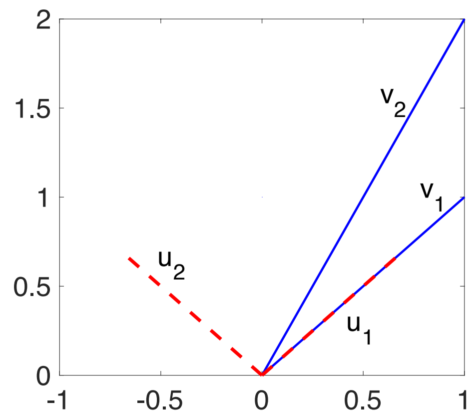
Gram-Schmidt Procedure

$\{v_1, v_2, \ldots, v_q\}$ is linearly independent.

Find an orthonormal basis

$$\{u_1, \ldots, u_q\}$$

for $S = \text{span}\{v_1, \ldots, v_q\}$.



$S_2 = \text{span}\{v_1, v_2\}$

Find u_1

$u_1 = v_1/\|v_1\|$

Find u_2

$v_2 = \alpha_1 u_1 + \alpha_2 u_2 \quad \Longrightarrow \quad \alpha_1 = \langle u_1, v_2\rangle$

$u_2 = \widetilde{u}_2/\|\widetilde{u}_2\|$, **where** $\qquad \widetilde{u}_2 = v_2 - \langle u_1, v_2\rangle u_1$

$S_j = \text{span}\{v_1, \ldots, v_j\}$

already computed

$\{u_1, \ldots, u_{j-1}\}$ orthonormal basis for $S_{j-1} = \text{span}\{v_1, \ldots, v_{j-1}\}$

Find u_j

$$v_j = \alpha_1 u_1 + \cdots + \alpha_{j-1} u_{j-1} + \alpha_j u_j \qquad \Longrightarrow$$
$$\alpha_k = \langle u_k, v_j \rangle \quad k = 1, \ldots, j-1$$

$$u_j = \widetilde{u}_j / \|\widetilde{u}_j\|, \quad \text{where} \qquad \widetilde{u}_j = v_j - \sum_{k=1}^{j-1} \langle u_k, v_j \rangle u_k$$

Gram-Schmidt, Implementation in $\mathbb{C}^n$

for $j = 1, \ldots, q$ **do**
    $u_j \longleftarrow v_j$
    for $k = 1, \ldots, j-1$ **do**
        $u_j \longleftarrow u_j - (u_k^* v_j) u_k$
    end for
    $u_j \longleftarrow u_j / \|u_j\|$
end for

Eigenvalues

A scalar $\lambda \in \mathbb{C}^n$ is an eigenvalue of $\mathbb{C}^{n \times n}$ if

$$Av = \lambda v \quad \exists v \in \mathbb{C}^n, \;\; v \neq 0. \tag{0.1}$$

- Nonzero v in (0.1) is an eigenvector corresponding to λ.

Hand calculation

$$\begin{aligned} Av = \lambda v \; \exists v \neq 0 \quad &\iff \quad (A - \lambda I)v = 0 \; \exists v \neq 0 \\ &\iff \quad \underbrace{\text{det}(A - \lambda I)}_{p(\lambda)} = 0 \end{aligned}$$

- $p(\lambda)$ is the characteristic polynomial of A.

Example

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad \text{det}\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{det}\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right)$$
$$= (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$$

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 2$

Eigenvector corresponding to $\lambda_1 = 4$

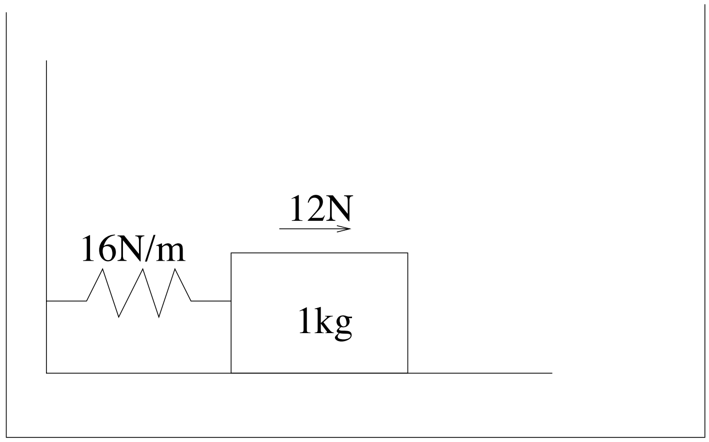
$$(A - 4I)v_1 = 0, \qquad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_1 = 0$$

Hence, $v_1 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for any nonzero scalar c_1.

Eigenvector corresponding to $\lambda_2 = 2$

$$(A - 2I)v_2 = 0, \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_2 = 0$$

$v_2 = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for any nonzero scalar c_2.



A Motivating Example

$x(t)$ is the displacement.

Friction in the ground, a force of $-kx'(t)$,
where k is the friction constant.

$$x''(t) = -kx'(t) - 16x(t)$$

Letting $y_1(t) := x(t), \quad y_2(t) := x'(t)$

$$\begin{aligned} y_2'(t) &= -ky_2(t) - 16y_1(t) \\ y_1'(t) &= y_2(t) \end{aligned}$$

that is

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -16 & -k \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

$$y'(t) = \underbrace{A}_{n\times n} y(t)$$

Eigenvalues of A: $\lambda_1, \ldots, \lambda_n$
Corresponding eigenvectors: $v_1, \ldots, v_n$

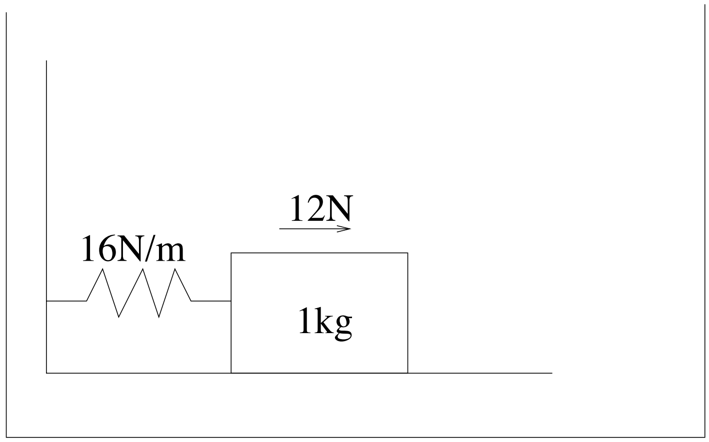
Solution

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

where $c_1, \ldots, c_n$ are constants to be determined using $y(0)$.

Verify

$$\begin{aligned}
(c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n)' &= (c_1 \lambda_1 e^{\lambda_1 t} v_1 + \cdots + c_n \lambda_n e^{\lambda_n t} v_n) \\
&= (c_1 e^{\lambda_1 t} A v_1 + \cdots + c_n e^{\lambda_n t} A v_n) \\
&= A(c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n)
\end{aligned}$$



$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -16 & -k \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Characteristic polynomial:
$p(\lambda) = (-\lambda)(-k-\lambda) + 16 = \lambda^2 + k\lambda + 16$

Eigenvalues: $\lambda_1 = \frac{-k+\sqrt{k^2-64}}{2}, \quad \lambda_2 = \frac{-k-\sqrt{k^2-64}}{2}$

Solution:

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

where c_1, c_2 satisfies

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 v_1 + c_2 v_2 = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$