Schur Factorization and PSD Hermitian Matrices

September 27, 2018

The Schur factorization for $A \in \mathbb{C}^{n\times n}$ is of the form

$$A = QTQ^*$$

where

- $Q \in \mathbb{C}^{n\times n}$ is unitary,
- $T \in \mathbb{C}^{n\times n}$ is upper triangular.

Example.

$$\underbrace{\begin{bmatrix} 6 & 2 \\ -4 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 4 & 6 \\ 0 & 2 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{Q^*}$$

Theorem 0.1 (Existence)**.** *Every matrix* $A \in \mathbb{C}^{n\times n}$ *has a Schur factorization.*

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Inductive case
Let λ be an eigenvalue and $v \in \mathbb{C}^n$ be a corresponding eigenvector. Form a unitary matrix

$$\widetilde{Q} = \left[\begin{array}{cc} v & \widetilde{\widetilde{Q}} \end{array} \right], \qquad \text{where} \quad \widetilde{\widetilde{Q}} = \left[\begin{array}{ccc} q_2 & \dots & q_n \end{array} \right].$$

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Observe

$$\begin{aligned} \widetilde{Q}^* A \widetilde{Q} &= \widetilde{Q}^* \begin{bmatrix} \lambda v & A\widetilde{\widetilde{Q}} \end{bmatrix} \\ &= \begin{bmatrix} \lambda \widetilde{Q}^* v & \widetilde{Q}^* A \widetilde{\widetilde{Q}} \end{bmatrix} = \begin{bmatrix} \lambda & w^T \\ 0 & \widehat{A} \end{bmatrix} \end{aligned}$$

for some $w \in \mathbb{C}^n$ and $\widehat{A} \in \mathbb{C}^{(n-1)\times(n-1)}$.

As the inductive hypothesis $\widehat{A}$ has a Schur factorization

$$\widehat{A} \;=\; \widehat{Q}\widehat{T}\widehat{Q}^*$$

where $\widehat{T} \in \mathbb{C}^{(n-1)\times(n-1)}$ is upper triangular,
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It follows that

$$\widetilde{Q}^* A \widetilde{Q} \;=\; \begin{bmatrix} \lambda & w^T \\ 0 & \widehat{Q}\widehat{T}\widehat{Q}^* \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & \widehat{Q} \end{bmatrix} \begin{bmatrix} \lambda & w^T\widehat{Q} \\ 0 & \widehat{T} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \widehat{Q}^* \end{bmatrix}$$

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that is

$$A = \underbrace{\widetilde{Q} \begin{bmatrix} 1 & 0 \\ 0 & \widehat{Q} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \lambda & w^T \widehat{Q} \\ 0 & \widehat{T} \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \widehat{Q}^* \end{bmatrix} \widetilde{Q}^*}_{Q^*}$$

as desired.

Hermitian Positive Semi-Definite (PSD) Matrices

Let $A \in \mathbb{C}^{n\times n},\ A^* = A$ and

$$v^*Av \geq 0 \qquad \forall v \in \mathbb{C}^n.$$

Theorem 0.2. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. The following are equivalent:*

(i) *A is positive semi-definite.*

(ii) *All eigenvalues of A are nonnegative.*

Example.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 4$, $\lambda = 2$.

A is positive semi-definite.

Proof of $\sim$(ii) $\Longrightarrow$ $\sim$(i)

Suppose A has a negative eigenvalue λ with a corresponding unit eigenvector v.

$$v^* A v \quad = \quad v^* \lambda v \quad = \quad \lambda \|v\|^2 \quad = \quad \lambda < 0$$

Hence, A is not positive semi-definite.

Proof of (ii) $\Longrightarrow$ (i)

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A, all nonnegative.
$v_1, \ldots, v_n$ be corresponding eigenvectors s.t.
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Every $v \in \mathbb{C}^n$ can be written of the form

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

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$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, and

$$\begin{aligned} v^* A v &= (\overline{\alpha}_1 v_1^* + \cdots + \overline{\alpha}_n v_n^*)(\alpha_1 \lambda_1 v_1 + \cdots + \alpha_n \lambda_n v_n) \\ &= |\alpha_1|^2 \lambda_1 v_1^* v_1 + \cdots + |\alpha_n|^2 \lambda_n v_n^* v_n \\ &= |\alpha_1|^2 \lambda_1 + \cdots + |\alpha_n|^2 \lambda_n \geq 0. \end{aligned}$$

Hence, A is positive semi-definite.