# Reminders from Lecture 4

# Singular Value Decomposition (SVD)

$A \in \mathbb{C}^{m \times n}, \;\; m \geq n$

Singular values of $A$,

$\sigma_1 \;\geq\; \sigma_2 \;\geq \ldots \;\geq\; \sigma_n \;\geq\; 0$

Square-roots of eigenvalues of $A^*A$

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Square-roots of eigenvalues of $A^*A$

Right singular vectors of $A$
$v_1, v_2, \dots, v_n \in \mathbb{C}^n$

$$A^*Av_j \;=\; \sigma_j^2 v_j \quad j = 1, \dots, n$$

and $\{v_1, \dots, v_n\}$ is orthonormal.

# Singular Value Decomposition

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Left singular vectors of $A$
$u_1, u_2, \ldots, u_n \in \mathbb{C}^n$

$\sigma_1, \ldots, \sigma_r > 0, \sigma_{r+1} = \cdots = \sigma_n = 0$

$$Av_j \ = \ \sigma_j u_j \quad j = 1, \ldots, r,$$

$u_{r+1}, \ldots, u_n \in \mathrm{Col}(A)^{\perp}$ s.t. $\{u_{r+1}, \ldots, u_n\}$ is orthonormal.

# Singular Value Decomposition

**Example** $A = \begin{bmatrix} -2 & 4 \\ -4 & 2 \end{bmatrix}, \qquad A^*A = \begin{bmatrix} 20 & -16 \\ -16 & 20 \end{bmatrix}$

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<u>Singular values</u>

$$\det(A^*A - \lambda I) = (20-\lambda)^2 - 256$$

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<u>Singular values</u>

$$\det(A^*A - \lambda I) = (20-\lambda)^2 - 256 = \lambda^2 - 40\lambda + 144 = (\lambda - 36)(\lambda - 4)$$

$\sigma_1 = 6$, $\sigma_2 = 2$

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$\sigma_1 = 6$, $\sigma_2 = 2$

Right singular vectors

$$\sigma_1 = 6, \quad (A^*A - 36I)v_1 = \begin{bmatrix} -16 & -16 \\ -16 & -16 \end{bmatrix} v_1 = 0$$

$$\sigma_2 = 2, \quad (A^*A - 4I)v_2 = \begin{bmatrix} 16 & -16 \\ -16 & 16 \end{bmatrix} v_2 = 0$$

$$\{v_1, v_2\} = \left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

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<u>Left singular vectors</u>

$$\sigma_1 = 6, \quad u_1 = Av_1/\sigma_1$$

$$\sigma_2 = 2, \quad u_2 = Av_2/\sigma_2$$

# Singular Value Decomposition

**Example** $A = \begin{bmatrix} -2 & 4 \\ -4 & 2 \end{bmatrix}, \qquad A^*A = \begin{bmatrix} 20 & -16 \\ -16 & 20 \end{bmatrix}$

Left singular vectors

$$\sigma_1 = 6, \quad u_1 = Av_1/\sigma_1 = (1/6)\begin{bmatrix} -2 & 4 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\sigma_2 = 2, \quad u_2 = Av_2/\sigma_2 = (1/2)\begin{bmatrix} -2 & 4 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\{u_1, u_2\} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

# Singular Value Decomposition

$$\begin{bmatrix} Av_1 & Av_2 & \dots & Av_n \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \dots & \sigma_n u_n \end{bmatrix}$$

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$$A \underbrace{\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}}_{\widehat{V} \in \mathbb{R}^{n \times n}} = \underbrace{\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}}_{\widehat{U} \in \mathbb{R}^{m \times n}} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}}_{\Sigma \in \mathbb{R}^{n \times n}}$$

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**Reduced SVD**

$$A = \widehat{U}\widehat{\Sigma}\widehat{V}^*$$

# Singular Value Decomposition

$$A = \begin{bmatrix} -2 & 4 \\ -4 & 2 \end{bmatrix}, \qquad \sigma_1 = 6,\ \sigma_2 = 2$$

$$\{v_1, v_2\} = \left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\},$$

$$\{u_1, u_2\} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

SVD

$$\underbrace{\begin{bmatrix} -2 & 4 \\ -4 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{\widehat{U}} \underbrace{\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}}_{\widehat{\Sigma}} \underbrace{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{\widehat{V}^*}$$