Reminders from Lecture 5

Outer Product Form of an SVD

Reduced SVD for an $A \in \mathbb{C}^{m \times n}$

$$A = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix}$$

Outer product form

$$A = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots + \sigma_n u_n v_n^*$$

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Outer product form

$$A = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots + \sigma_n u_n v_n^*$$

Suppose $\sigma_1, \dots, \sigma_r \gg \sigma_{r+1}, \dots, \sigma_n$, then

$$A \approx \sigma_1 u_1 v_1^* + \cdots + \sigma_r u_r v_r^*.$$

Outer Product Form of an SVD

$$\begin{bmatrix} 11 & 9 \\ 10 & 10 \\ 9 & 11 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{6} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= (10\sqrt{6}) \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 2 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

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$$= \begin{bmatrix} 10 & 10 \\ 10 & 10 \\ 10 & 10 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$$

Full SVD

Reduced SVD for an $A \in \mathbb{C}^{m \times n}$ (with $m \geq n$)

$$A = \underbrace{\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}}_{\widetilde{U} \in \mathbb{C}^{m \times n}} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}}_{\widetilde{\Sigma} \in \mathbb{R}^{n \times n}} \underbrace{\begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix}}_{\widetilde{V}^* \in \mathbb{C}^{n \times n}}$$

Full singular value decomposition

$$A = \underbrace{\begin{bmatrix} \widetilde{U} & \widehat{U} \end{bmatrix}}_{U \in \mathbb{C}^{m \times m}} \underbrace{\begin{bmatrix} \widetilde{\Sigma} \\ 0 \end{bmatrix}}_{\Sigma \in \mathbb{R}^{m \times n}} \underbrace{\widetilde{V}^*}_{V^*}$$

U, V - unitary

Σ - diagonal with nonnegative real entries

Full SVD

Reduced SVD

$$\begin{bmatrix} 11 & 9 \\ 10 & 10 \\ 9 & 11 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{6} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Full SVD

$$\begin{bmatrix} 11 & 9 \\ 10 & 10 \\ 9 & 11 \end{bmatrix} =$$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 10\sqrt{6} & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Matrix 2-Norm

For an $A \in \mathbb{C}^{m \times n}$

$$\|A\|_2 \quad := \quad \max_{v \in \mathbb{C}^n,\, \|v\|_2 = 1} \|Av\|_2$$

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$$\|A\|_2 \quad = \quad \sigma_1$$

e.g.,

$$\begin{bmatrix} 11 & 9 \\ 10 & 10 \\ 9 & 11 \end{bmatrix} \qquad \sigma_1 = 10\sqrt{6}, \sigma_2 = 2$$

$\|A\|_2 = 10\sqrt{6}$