Best Low-Rank Approximation Problem

October 9, 2018

Problem Statement.
Let $A \in \mathbb{C}^{m\times n}$ and p be a positive integer.

Find $B_* \in \mathbb{C}^{m\times n}$ such that $\mathrm{rank}(B_*) \le p$, and

$$\|B_* - A\|_2 \le \|B - A\|_2$$

for all $B \in \mathbb{C}^{m\times n}$ with $\mathrm{rank}(B) \le p$.

SVD for A in outer product form

$$A = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^* + \ldots + \sigma_p u_p v_p^* + \ldots + \sigma_\ell u_\ell v_\ell^*, \qquad \ell := \min\{m, n\}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\ell$$

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$$\widehat{B} = \sigma_1 u_1 v_1^* \ + \ \sigma_2 u_2 v_2^* \ + \ \ldots \ + \ \sigma_p u_p v_p^*$$

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$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\ell$$

Consider

$$\widehat{B} = \sigma_1 u_1 v_1^* \; + \; \sigma_2 u_2 v_2^* \; + \; \dots \; + \; \sigma_p u_p v_p^*$$

- rank$(\widehat{B}) \leq p$
- $\|\widehat{B} - A\|_2 \;\; = \;\; \| - \sigma_{p+1} u_{p+1} v_{p+1}^* \; - \; \dots \; - \; \sigma_\ell u_\ell v_\ell^* \|_2 \;\; = \;\; \sigma_{p+1}$

Example. $p = 1$ and

$$\underbrace{\begin{bmatrix} 11 & 9 \\ 10 & 10 \\ 9 & 11 \end{bmatrix}}_{A} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{6} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \quad (10\sqrt{6}) \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 2 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

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$$\widehat{B} = (10\sqrt{6}) \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 10 & 10 \\ 10 & 10 \end{bmatrix}, \qquad \|\widehat{B} - A\|_2 \;=\; 2$$

Theorem 0.1. *For all* $B \in \mathbb{C}^{m\times n}$ *such that* $\operatorname{rank}(B) \leq p$*,*

$$\|B - A\|_2 \quad \geq \quad \sigma_{p+1}.$$

Proof.

The dimension of the null space of B defined by

$$\text{Null}(B) := \{w \in \mathbb{C}^n \mid Bw = 0\}$$

is at least $n - p$.

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is at least $n - p$.

Letting $\mathcal{S} := \mathrm{span}\{v_1, \ldots, v_p, v_{p+1}\}$, we have

$$\text{Null}(B) \cap \mathcal{S} \neq \{0\}.$$

Proof.

Take any unit vector $v \in \text{Null}(B) \cap \mathcal{S}$.

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Take any unit vector $v \in \text{Null}(B) \cap \mathcal{S}$. But then

$$\|B - A\|_2 \;\; \geq \;\; \|(B - A)v\|_2 \;\; = \;\; \|Av\|_2 \;\; \geq \;\; \sigma_{p+1}$$

completing the proof.

Theorem 0.2 (Eckart-Young)**.** *Let* $A \in \mathbb{C}^{m \times n}$*, and* p *be a positive integer. Then*

$$\min_{\substack{B \in \mathbb{C}^{m \times n} \\ \textit{rank}(B) \leq p}} \|B - A\|_2 \quad = \quad \sigma_{p+1} \quad = \quad \|B_* - A\|_2$$

where

$$B_* \;\; := \;\; \sigma_1 u_1 v_1^* \; + \; \sigma_2 u_2 v_2^* \; + \; \ldots \; + \; \sigma_p u_p v_p^*.$$

Theorem 0.3 (Eckart-Young)**.** *Let* $A \in \mathbb{C}^{m\times n}$*, and* p *be a positive integer. Then*

$$\min_{\substack{B \in \mathbb{C}^{m\times n} \\ \textit{rank}(B) \le p}} \|B - A\|_2 \quad = \quad \sigma_{p+1} \quad = \quad \|B_* - A\|_2$$

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Example. $p = 1$

$$A \; = \; \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} \; = \; \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Theorem 0.4 (Eckart-Young)**.** *Let $A \in \mathbb{C}^{m \times n}$, and p be a positive integer. Then*

$$\min_{\substack{B \in \mathbb{C}^{m\times n} \\ \textit{rank}(B) \leq p}} \|B - A\|_2 \quad = \quad \sigma_{p+1} \quad = \quad \|B_* - A\|_2$$

where

$$B_* \; := \; \sigma_1 u_1 v_1^* \; + \; \sigma_2 u_2 v_2^* \; + \; \ldots \; + \; \sigma_p u_p v_p^*.$$

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$$B_* \;=\; 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \;=\; \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$