Reminders from Lecture 8 & Background

Direct Sum

For two subspaces of $\mathcal{S}_1, \mathcal{S}_2$ of $\mathbb{C}^n$ such that $\mathcal{S}_1 \cap \mathcal{S}_2 = \{0\}$, their direct sum is defined by

$$\mathcal{S}_1 \oplus \mathcal{S}_2 \quad := \quad \{v_1 + v_2 \mid v_1 \in \mathcal{S}_1,\ v_2 \in \mathcal{S}_2\}$$

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- $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \oplus \operatorname{span}\left\{\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}\right\}$ is not defined.

Projections

Let $\mathcal{S}_1, \mathcal{S}_2$ be subspaces s.t. $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathbb{C}^n$.
Every vector $v \in \mathbb{C}^n$ has a unique decomposition

$$v = v_{\mathcal{S}_1} + v_{\mathcal{S}_2}, \quad \exists v_{\mathcal{S}_1} \in \mathcal{S}_1, \; \exists v_{\mathcal{S}_2} \in \mathcal{S}_2.$$

$v_{\mathcal{S}_1}$ is called the projection of v onto $\mathcal{S}_1$ along $\mathcal{S}_2$.

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Example. $\mathcal{S}_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \mathcal{S}_2 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

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$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = (5) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$v_{\mathcal{S}_1} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is the projection of v onto $\mathcal{S}_1$ along $\mathcal{S}_2$.

Orthogonal Complement

Given $\mathcal{S}$ a subspace of $\mathbb{C}^n$, orthogonal complement of $\mathcal{S}$

$$\mathcal{S}^\perp \;:=\; \{w \in \mathbb{C}^n \mid w^* v = 0 \;\text{ for all } v \in \mathcal{S}\}$$

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$$= \left\{ \begin{bmatrix} x \\ y \\ -x - y \end{bmatrix} \;\middle|\; x, y \in \mathbb{C} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Orthogonal Complement

$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \end{bmatrix} \text{ is unitary.}$$

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- $U = \begin{bmatrix} \widehat{U} & \widetilde{U} \end{bmatrix} \in \mathbb{C}^{n \times n}$ is unitary, $\widehat{U} \in \mathbb{C}^{n \times p}$, $\widetilde{U} \in \mathbb{C}^{n \times (n-p)}$

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