MATH 504: Numerical Methods I

Instructor: Emre Mengi

Fall Semester 2018
Final Examination

NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

#1	20	
#2	20	
#3	25	
#4	20	
#5	15	
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- Put your name, student ID and signature in the spaces provided above.
- Duration for this exam is 165 minutes.

Problem 1. (20 points) Let $A \in \mathbb{C}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$ such that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n-1}| > |\lambda_n|,$$

and n linearly independent eigenvectors $v_1, v_2, \ldots, v_{n-1}, v_n$, where v_j is an eigenvector corresponding to λ_j for $j = 1, \ldots, n$.

Suppose also that an LU factorization of $A \in \mathbb{C}^{n \times n}$ obtained by employing partial pivoting strategy is also given, that is a permutation matrix $P \in \mathbb{R}^{n \times n}$, a unit lower triangular matrix $L \in \mathbb{C}^{n \times n}$, and an upper triangular matrix $U \in \mathbb{C}^{n \times n}$ satisfying

$$PA = LU$$

are given.

Given A as above and $q^{(0)} \in \mathbb{C}^n$ such that $q^{(0)} \notin \text{span}\{v_1, \ldots, v_{n-1}\}$, write down a pseudocode that generates a sequence $\{q^{(k)}\}$ in $\mathbb{C}^n$ satisfying

$$\text{span}\{q^{(k)}\} \to \text{span}\{v_n\} \qquad \text{as} \quad k \to \infty.$$

Your pseudocode must perform as few flops as possible.

Problem 2. Let

$$A = \begin{bmatrix} -2 & -4 & 3 \\ 4 & -1 & -6 \\ 1 & 2 & 3 \end{bmatrix}.$$

(a) **(10 points)** Compute a unit lower triangular matrix $L \in \mathbb{R}^{3\times 3}$ (that is L must be a lower triangular matrix with 1s on the diagonal) and an upper triangular matrix $U \in \mathbb{R}^{3\times 3}$ such that

$$A = LU.$$

(b) **(10 points)** Compute a unit upper triangular matrix $U \in \mathbb{R}^{3\times 3}$ (that is U must be an upper triangular matrix with 1s on the diagonal) and a lower triangular matrix $L \in \mathbb{R}^{3\times 3}$ such that

$$A = UL.$$

Problem 3. (25 points) For every matrix $A \in \mathbb{C}^{n\times n}$, there exist unitary matrices $U, V \in \mathbb{C}^{n\times n}$ such that

$$UAV = B \tag{1}$$

is bidiagonal (that is B is such that $b_{ij} = 0$ if $j \neq i$ and $j \neq i+1$).

Write down a pseudocode that, for a given $A \in \mathbb{C}^{n\times n}$, computes a bidiagonal matrix $B \in \mathbb{C}^{n\times n}$ such that (1) holds for some unitary matrices $U, V \in \mathbb{C}^{n\times n}$. Your pseudocode does not have to return U and V, it suffices if it only returns B.

Problem 4. The Hadamard product $\odot$ of two matrices $A, B \in \mathbb{C}^{n\times n}$ is defined by

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}}_{A} \odot \underbrace{\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & & b_{nn} \end{bmatrix}}_{B} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \dots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & & a_{2n}b_{2n} \\ \vdots & & & \\ a_{n1}b_{n1} & a_{n2}b_{n2} & & a_{nn}b_{nn} \end{bmatrix}.$$

For a given $D \in \mathbb{C}^{n\times n}$, define

$$f : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, \quad f(X) \;=\; D \odot X. \tag{2}$$

(a) **(10 points)** Assuming D and X in (2) have exact representations in IEEE floating point arithmetic, show that $\widehat{f}(X)$, that is the computed $f(X)$ in IEEE floating point arithmetic, satisfies

$$\widehat{f}(X) \;=\; f(X + \delta X), \qquad \exists\, \delta X \in \mathbb{C}^{n\times n} \text{ s.t. } \|\delta X\|_F / \|X\|_F = O(\epsilon_{\text{mach}}).$$

(b) **(10 points)** Let κ denote the absolute condition number of $f(X)$ defined as in (2) for a given $D \in \mathbb{C}^{n\times n}$ when the Frobenius norm $\|\cdot\|_F$ is used on the input and output spaces of $f(X)$. Show that $\kappa = \|D\|_F$.

(Note: You can make use of the inequality

$$\|A \odot B\|_F \leq \|A\|_F \|B\|_F$$

that is satisfied for every $A, B \in \mathbb{C}^{n\times n}$*.)*

Problem 5. (15 points) A pseudocode is provided below for the basic QR algorithm without shifts to compute the eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$.

Algorithm 1 The QR Algorithm without Shifts

$A_0 \leftarrow A$
for $k = 0, 1, \ldots$ **do**
  Compute a QR factorization $A_k = Q_{k+1}R_{k+1}$
  $A_{k+1} \leftarrow R_{k+1}Q_{k+1}$
end for

Prove that the iterates Q_k, R_k by this algorithm satisfy

$$A^k = (Q_1 Q_2 \ldots Q_k)(R_k \ldots R_2 R_1)$$

for all integer $k \geq 1$.