MATH 504: Numerical Methods I

Instructor: Emre Mengi

Fall Semester 2016
Final Exam
Friday January 6th, 2017

NAME ____________________

SIGNATURE ____________________

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#2	15	
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- Please put your name and signature in the space provided above.
- This is a closed-book and closed-notes exam.
- You are expected to support your answer in each question, or otherwise you may not be awarded full-credit.

Question 1 (15 points) Solve the following least squares problem:

$$\text{minimize}_{X\in\mathbb{C}^{2\times 2}} \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} X \right\|_F .$$

Question 2 (15 points)
Let $A \in \mathbb{C}^{n\times n}$ be a given matrix, $\lambda \in \mathbb{C}$ be a given scalar, r be a given positive integer. Write down a matrix $\delta A \in \mathbb{C}^{n\times n}$ such that

(1) $\|\delta A\|_2$ is as small as possible, and

(2) λ is an eigenvalue of $A + \delta A$ with geometric multiplicity greater than or equal to r.

Explain your answer.

Question 3

Let us consider a matrix $A \in \mathbb{C}^{m\times n}$ with SVD

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}$$

where $\Sigma_1 \in \mathbb{R}^{r\times r}$ is diagonal with positive diagonal entries, $U_1 \in \mathbb{C}^{m\times r}$, $U_2 \in \mathbb{C}^{m\times(m-r)}$, $V_1 \in \mathbb{C}^{n\times r}$ and $V_2 \in \mathbb{C}^{n\times(n-r)}$.

(a) (10 points) Write down expressions for the orthogonal projectors onto $\text{Col}(A)$ and $\text{Null}(A)^\perp$ in terms of U_1, U_2, V_1, V_2. (The expressions can depend on some or all of U_1, U_2, V_1, V_2.)

(b) (5 points) Write down the orthogonal eigenvalue decomposition of A^*A in terms of $\Sigma_1, U_1, U_2, V_1, V_2$. (Once again your expression can depend on some or all of these matrices.)

Question 4 Let A and B be the 20×20 symmetric Toeplitz matrices such that $a_{jj} = 1$, $b_{jj} = 3$ for $j = 1, \ldots, 20$, and $a_{ij} = b_{ij} = 2^{-|i-j|-1}$ for $i, j = 1, \ldots, 20$ such that $i \neq j$. Consider the linear systems

$$Ax = b \quad \text{and} \quad Bx = b$$

for $b \in \mathbb{R}^{20}$ equal to the vector of ones.

(a) (10 points) Would you expect in theory that *Richardson iteration* converge for each one of these two linear systems? Explain.

(b) (5 points) Does *Jacobi iteration* converge for each one of these two linear systems? Explain.

Question 5 (20 points) For a given matrix $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, write down a pseudocode that generates a bidiagonal matrix $B \in \mathbb{C}^{m\times n}$ such that

$$B \; = \; UAV$$

for some unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$. This is the first stage in singular value computation, and reduces the computational cost in the second stage immensely.

Your pseudocode should only return the bidiagonal matrix B, in particular it does not have to form or return the unitary matrices U and V. Make sure that your pseudocode requires as few flops as possible.
(Note: Recall that a matrix B is bidiagonal if $b_{jk} = 0$ unless $k = j$ or $k = j+1$.)

Question 6 (20 points) Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Furthermore, let $v^{(j)}$ denote an eigenvector of A that corresponds to the jth largest eigenvalue of A in absolute value.

Write down an algorithm that generates a sequence of matrices $\{Q_k\}$ in $\mathbb{C}^{n\times 3}$ such that, under mild conditions,

$$\mathrm{Col}(Q_k(:,j)) \;\rightarrow\; \mathrm{span}\left\{v^{(j)}\right\} \qquad \text{as } k \rightarrow \infty$$

for $j = 1, 2, 3$. State also precisely the mild conditions that ensure the convergence.