MATH 504: Numerical Methods I

Instructor: Emre Mengi

Fall Semester 2018
1st Midterm Examination

#1	30	
#2	25	
#3	25	
#4	20	
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NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

- Put your name, student ID and signature in the spaces provided above.
- Duration for this exam is 120 minutes.

Problem 1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

(a) **(20 points)** Calculate a reduced singular value decomposition of A given in (1).

(b) **(10 points)** For the matrix A in (1), write down a 3×2 matrix B_* such that $\text{rank}(B_*) = 1$ and

$$\|B_* - A\|_2 \leq \|B - A\|_2$$

for all $B \in \mathbb{C}^{3\times 2}$ with $\text{rank}(B) = 1$.

Problem 2. Consider the 3×3 matrix given below.

$$A = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \quad (2)$$

(a) **(10 points)** Is A in (2) positive semi-definite? If it is, explain why. If it is not, provide a vector $q \in \mathbb{R}^3$ such that $q^T A q < 0$.

(b) **(15 points)** Write down a full singular value decomposition of A in (2). Describe also what the set

$$\{Av \mid v \in \mathbb{R}^3 \text{ such that } \|v\|_2 = 1\}$$

corresponds to geometrically.

Problem 3. Suppose that $P \in \mathbb{C}^{n\times n}$ is a projector, but $P \neq I_n$ and $P \neq 0$.

- **(a)** **(9 points)** Show that $\|P\|_2 = 1$.
- **(b)** **(8 points)** Show that $\sigma_n(P) = 0$,
  that is show that the smallest singular value of P is 0.
- **(c)** **(8 points)** Now suppose in particular that P is the orthogonal projector onto span$\{v\}^\perp$ for a given nonzero vector $v \in \mathbb{C}^n$. Give an expression for P in terms of v.

Problem 4. (20 points) A matrix $H \in \mathbb{C}^{n\times n}$ is called Hessenberg if $h_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ such that $i - j > 1$. For instance, the matrix

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ -4 & 1 & 2 & 1 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 2 & -5 \end{bmatrix}$$

is Hessenberg.

For every matrix $A \in \mathbb{C}^{n\times n}$, there exists a unitary matrix $Q \in \mathbb{C}^{n\times n}$ and a Hessenberg matrix $H \in \mathbb{C}^{n\times n}$ such that

$$Q^* A Q \quad = \quad H. \tag{3}$$

Given a matrix $A \in \mathbb{C}^{n\times n}$, design an efficient algorithm, in particular write down a pseudocode, that computes a Hessenberg matrix $H \in \mathbb{C}^{n\times n}$ satisfying (3) for some unitary matrix $Q \in \mathbb{C}^{n\times n}$. Your algorithm does not need to form the unitary matrix Q explicitly, rather it suffices if it only returns H. Count the number of flops required by your algorithm.