Math 504: Numerical Methods - I

Midterm - Fall 2010

Duration : 75 minutes

#1	20	
#2	20	
#3	20	
#4	20	
#5	20	
Σ	100	

Name ____________________

Student ID ____________________

Signature ____________________

- Put your name and student ID in the boxes above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.

Question 1. Consider the matrix

$$A = \begin{bmatrix} 1 & 9 \\ 12 & 8 \end{bmatrix}$$

with 2-norm equal to $5\sqrt{10}$. Note also that

$$A \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 5\sqrt{10} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

(a) Find a singular value decomposition for A.

(b) Calculate

(i) the distance to the nearest singular matrix in 2-norm, that is calculate

$$\beta(A) = \min\{\|\Delta A\|_2 : (A + \Delta A) \text{ is singular}\},$$

(ii) the nearest singular matrix $(A + \Delta A_*)$ satisfying

$$\|\Delta A_*\|_2 = \beta(A).$$

Question 2. This question concerns the orthogonal projectors.

(a) Given the subspace $\mathcal{W} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

(i) Find an orthonormal basis for the subspace that is orthogonal to $\mathcal{W}$.

(ii) Find an orthogonal projector onto $\mathcal{W}$.

(b) Show that if $P \in \mathbf{C}^{n \times n}$ is an orthogonal projector and $v \in \mathbf{C}^n$, then Pv is the orthogonal projection of v onto range(P). Use the definition of an orthogonal projection given below in your proof.

> **<u>Definition:</u>** Given a subspace $\mathcal{S}$ of $\mathbf{C}^n$ and a vector $v \in \mathbf{C}^n$. We call the vector $v_{\mathcal{S}} \in \mathbf{C}^n$ the orthogonal projection of v onto $\mathcal{S}$ if
>
> $$\textbf{(i)} v_{\mathcal{S}} \in \mathcal{S} \quad \text{and} \quad \textbf{(ii)} v_{\mathcal{S}} \perp (v - v_{\mathcal{S}}).$$

Question 3. A matrix $H \in \mathbf{C}^{m \times n}$ with $m \geq n$ is called Hessenberg if $h_{ij} = 0$ whenever $i > (j + 1)$. Every matrix $A \in \mathbf{C}^{m \times n}$ has a QH factorization of the form

$$A = QH$$

where $Q \in \mathbf{C}^{m \times m}$ is unitary and $H \in \mathbf{C}^{m \times n}$ is Hessenberg.

Devise an algorithm to calculate the QH factorization of $A \in \mathbf{C}^{m \times n}$ and calculate the total number of flops required by your algorithm.

Question 4. The population of the US is listed at various years in the table below.

t (year)	$s = (t - 1940)/5$	y (population in ten millions)
1955	$s_1 = 3$	$y_1 = 17$
1960	$s_2 = 4$	$y_2 = 18$
2000	$s_3 = 12$	$y_3 = 28$

(a) Pose the problem of finding the line $\ell(s) = x_1 s + x_0$ minimizing

$$\sqrt{\sum_{i=1}^{3} (\ell(s_i) - y_i)^2}$$

with respect to the unknows $x_0, x_1 \in \mathbf{R}$ as a least-squares problem in the form $\boxed{\text{minimize}_{x \in \mathbf{R}^2} \|Ax - b\|_2}$.

(b) Solve the least squares problem in **(a)**, consequently determine the line that best represents the US population in the least-squares sense, by exploiting the QR factorization of A.

Question 5. Consider the roots of the polynomial $q(x) = x^2 + ax - 2$ as a function of $a \in \mathbf{R}$. In particular in this question you are expected to analyze the sensitivity of the root of $q(x)$ that is equal to 1 when $a = 1$ with respect to perturbations in a. Denote this root (as a function of a) by $r : \mathbf{R} \to \mathbf{R}$.

(a) Find the absolute condition number κ for r at $a = 1$.

(b) Find the relative condition number $\tilde{\kappa}$ for r at $a = 1$.