MATH 504: Numerical Methods - I

Midterm - Fall 2013
Duration : 100 minutes

#1	25	
#2	30	
#3	20	
#4	25	
Σ	100	

NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

- Put your name and student ID in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book exam, but you can use notes.
- Show all of your work; full credit will not be given for unsupported answers.

Question 1.

(a) *(15 pts)* Write down a rank two matrix $A \in \mathbb{R}^{3\times 2}$ with a pair of consistent left and right singular vectors given by

$$u = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

respectively. In other words, your matrix must satisfy $Av = \sigma u$ and $u^T A = \sigma v^T$ for some real scalar $\sigma > 0$.

(b) *(10 pts)* Find a matrix $B \in \mathbb{R}^{3\times 2}$ of rank one such that $\|B - A\|_2$ is as small as possible where A is the matrix you provided in part **(a)**.

Question 2.

(a) *(10 pts)* Let $A \in \mathbb{R}^{m\times n}$. Show that $\text{Col}(A^T) = \text{Null}(A)^{\perp}$.

(b) *(10 pts)* Calculate a QR factorization for B^T where

$$B = \begin{bmatrix} 2 & -2 & -2 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

(c) *(10 pts)* Determine the orthogonal projector onto $\text{Null}(B)$ for B in part **(b)**.

Question 3. This question concerns the following least squares problem:

$$\text{minimize}_{x\in\mathbb{R}^2} \left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} x \right\|$$

(a) *(10 pts)* Find a solution x_* to the least squares problem above.

(b) *(10 pts)* Find the solution x_{**} to least squares problem above such that $\|x_{**}\|_2 \leq \|\hat{x}\|_2$ for any solution $\hat{x}$ to the least squares problem.

Question 4. *(25 pts)* A matrix $H \in \mathbb{R}^{m\times n}$ is called *Hessenberg* if $h_{ij} = 0$ whenever $i > j + 1$. For instance a 4×4 Hessenberg matrix is of the form

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix},$$

where x denotes an entries that is possibly nonzero.

Given an Hessenberg matrix $H \in \mathbb{R}^{m\times n}$ with $m \geq n$, write down an algorithm to compute its QR factorization of the form $H = QR$ based on Householder reflectors, where $Q \in \mathbb{R}^{m\times m}$ is orthogonal and $R \in \mathbb{R}^{m\times n}$ is upper triangular.

You do not need to form the Q factor explicitly, it would suffice to return the vectors q_k such that $Q_k = I - 2q_k q_k^T$ is the Householder reflectors used at the k-th step. Make sure that your algorithm performs as few flops as possible. Count the number of flops required by your algorithm.