Math 504: Numerical Methods I

Instructor: Emre Mengi

Fall Semester 2016
Midterm Exam
Thursday November 10th, 2016

Name ____________________

Signature ____________________

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- Please put your name and signature in the space provided above.
- This is a closed-book and closed-notes exam.
- You are expected to support your answer in each question, or otherwise you may not be awarded full-credit.

Question 1 Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 2 & 6 \end{bmatrix}.$$

(a) (10 points) Compute a *reduced QR factorization* for A by means of the Gram-Schmidt procedure.

(b) (10 points) Compute a *full QR factorization* for A by means of the Householder triangularization.

Question 2
Let us consider the matrix A given below in its orthogonal rank 1 form.

$$A = 3\sqrt{6}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} + 2\sqrt{7}\begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

(a) (8 points) Write down the orthogonal projector onto $\mathrm{Col}(A)^{\perp}$.

(b) (7 points) Provide a matrix $B \in \mathbb{R}^{3\times 2}$ that is of rank 1, and such that $\|B - A\|_2$ is as small as possible.

(c) (7 points) Write down the pseudoinverse of A.

(d) (8 points) Find

- a basis $\mathcal{Y}$ for $\operatorname{Col}(A)$,
- a basis $\mathcal{X}$ for $\mathbb{C}^2$,
- and a diagonal matrix $\Sigma \in \mathbb{R}_+^{2\times 2}$ with nonnegative real entries along its diagonal

such that the following holds:

$$[y]_{\mathcal{Y}} = \Sigma [x]_{\mathcal{X}} \quad \forall x \in \mathbb{C}^2,\ \forall y \in \mathbb{C}^3 \text{ satisfying } y = Ax.$$

Question 3

(a) (15 points) Consider the linear system $Ax = b$ for a given invertible matrix $A \in \mathbb{C}^{n \times n}$ and a given vector $b \in \mathbb{C}^n$.

Prove that the *relative condition number* of the solution x of this linear system with respect to perturbations in A - keeping the right-hand side $b \in \mathbb{C}^n$ fixed, using the matrix 2-norm on $\mathbb{C}^{n \times n}$ and the vector 2-norm on $\mathbb{C}^n$ - is given by

$$\widetilde{\kappa} \;=\; \|A\|_2 \, \|A^{-1}\|_2 \, .$$

(b) (5 points) Now suppose that the linear system $Ax = b$ is solved for x by a backward stable algorithm. In particular, the computed solution $\widehat{x}$ by this algorithm satisfies

$$(A + \delta A)\widehat{x} = b \;\text{ for some }\; \delta A \in \mathbb{C}^{n\times n} \;\text{ such that }\; \frac{\|\delta A\|_2}{\|A\|_2} = O(\epsilon_{\text{mach}}).$$

Perform a *backward error analysis* to deduce an upper bound on the relative error

$$\frac{\|\widehat{x} - x\|_2}{\|x\|_2}\,.$$

Question 4

(a) (10 points) A matrix $A \in \mathbb{C}^{n \times p}$ with $n \geq p$ is called *upper Hessenberg* if $a_{jk} = 0$ whenever $j > k + 1$. For instance, a 4×3 upper Hessenberg matrix is of the form

$$\begin{bmatrix} x & x & x \\ x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$$

where an x denotes an entry that is possibly not zero.

Every matrix $A \in \mathbb{C}^{n \times p}$ with $n \geq p$ has a QH factorization of the form

$$A = Q \cdot H$$

where $Q \in \mathbb{C}^{n \times n}$ is unitary, and $H \in \mathbb{C}^{n \times p}$ is upper Hessenberg.

Write down a pseudocode based on Householder reflectors that computes a QH factorization for a given $A \in \mathbb{C}^{n \times p}$ with $n \geq p$.

- Your pseudocode must return H explicitly.
- It can return a set of vectors $u^{(1)}, u^{(2)}, \ldots$ (rather than the unitary factor Q), as the practice followed in the class, such that Q is defined implicitly in terms of $I - 2u^{(j)}[u^{(j)}]^*$.

(b) (15 points) Now suppose you are given a square matrix $A \in \mathbb{C}^{n \times n}$ such that $A(1:j, 1:j)$ is invertible for $j = 1, \ldots, n$. The invertibility condition implies the existence of a lower triangular matrix $L \in \mathbb{C}^{n \times n}$, an upper triangular matrix $U \in \mathbb{C}^{n \times n}$, and a diagonal matrix $D \in \mathbb{C}^{n \times n}$ such that

$$L \cdot A \cdot U = D \, . \tag{1}$$

Write down a pseudocode that computes a lower triangular $L \in \mathbb{C}^{n \times n}$, an upper triangular $U \in \mathbb{C}^{n \times n}$ and a diagonal $D \in \mathbb{C}^{n \times n}$ satisfying (1). Perform a flop count for your pseudocode.