Solutions

MATH 504: Numerical Methods I

Instructor: Emre Mengi

Fall Semester 2018
1st Midterm Examination

NAME ________________

STUDENT ID ________________

SIGNATURE ________________

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- Put your name, student ID and signature in the spaces provided above.
- Duration for this exam is 120 minutes.

Problem 1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

(a) **(20 points)** Calculate a reduced singular value decomposition of A given in (1).

(b) **(10 points)** For the matrix A in (1), write down a 3×2 matrix B_* such that $\text{rank}(B_*) = 1$ and

$$\|B_* - A\|_2 \;\leq\; \|B - A\|_2$$

for all $B \in \mathbb{C}^{3\times 2}$ with $\text{rank}(B) = 1$.

(a) Singular values σ_1, σ_2 are, square-roots of eigenvalues of $A^T A$

$$A^T A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\det(A^T A - \lambda I) = (5-\lambda)^2 - 16$$
$$= \lambda^2 - 10\lambda + 9$$

$\lambda_1 = 9$ and $\lambda_2 = 1$

$\Longrightarrow \sigma_1 = 3, \; \sigma_2 = 1$

Right singular vectors v_1, v_2 are eigenvectors of $A^T A$ corr. to σ_1^2, σ_2^2

(choose $c_1 = 1$)

$$\left(\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - 9I\right) v_1 = 0 \Longrightarrow v_1 = c_1 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \; \forall c_1 \neq 0$$

$$\left(\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - I\right) v_2 = 0 \Rightarrow v_2 = c_2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \; \forall c_2 \neq 0$$

(choose $c_2 = 1$)

Left singular vectors u_1, u_2

$$u_1 = Av_1/\sigma_1 = \begin{bmatrix} 1/3\sqrt{2} \\ 4/3\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix}$$

$$u_2 = Av_2/\sigma_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

The reduced SVD of A

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{array}{cc} u_1 & u_2 \\ \left[\begin{matrix} 1/3\sqrt{2} & 1/\sqrt{2} \\ 4/3\sqrt{2} & 0 \\ 1/3\sqrt{2} & -1/\sqrt{2} \end{matrix}\right. & \left.\right] \end{array}}_{\hat{U}} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{\hat{\Sigma}} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{array}{c} \to v_1^* \\ \to v_2^* \end{array}}_{\hat{V}^*}$$

$\sigma_1 = 3$, $\sigma_2 = 1$

(b) By low-rank approximation (Eckart-Young) thm nearest rank 1 matrix is given by

$$B_* = \sigma_1 u_1 v_1^*$$

$$= 3 \begin{bmatrix} 1/3\sqrt{2} \\ 4/3\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 2 & 2 \\ 1/2 & 1/2 \end{bmatrix}$$

Moreover,

$$\| B_* - A \|_2 = \sigma_2 = 1$$

Problem 2. Consider the 3×3 matrix given below.

$$A = \underbrace{\begin{array}{ccc} \uparrow q_1 & \uparrow q_2 & \uparrow q_3 \\ \end{array}\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 \leftarrow \lambda_1 & 0 & 0 \\ 0 & -2 \leftarrow \lambda_2 & 0 \\ 0 & 0 & -1 \leftarrow \lambda_3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}}_{Q^*} \quad (2)$$

(a) **(10 points)** Is A in (2) positive semi-definite? If it is, explain why. If it is not, provide a vector $q \in \mathbb{R}^3$ such that $q^T A q \leq 0$.

(b) **(15 points)** Write down a full singular value decomposition of A in (2). Describe also what the set

$$S = \{Av \mid v \in \mathbb{R}^3 \text{ such that } \|v\|_2 = 1\}$$

corresponds to geometrically.

(a) Notice

$$AQ = QD$$

$$\Longrightarrow Aq_j = \lambda_j q_j \quad j = 1, 2, 3$$

hence $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A

q_1, q_2, q_3 are corresponding eigenvectors

Since $\lambda_2, \lambda_3 < 0$, A is not positive semi-definite

Indeed, $q_2^T A q_2 = \begin{bmatrix} -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} A \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$

$$= \lambda_2 = -2.$$

(b) It suffices to negate λ_2, λ_3, to compensate this negate also q_2, q_3 in Q factor

$$A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

is an SVD of A. S is an ellipsoid in $\mathbb{R}^3$ with principal semi-axes $3\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $2\begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $1\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$.

Problem 3. Suppose that $P \in \mathbb{C}^{n\times n}$ is a projector, but $P \neq I_n$ and $P \neq 0$.

~~**(a) (9 points)** Show that~~ $\|P\|_2 = 1$.

(b) (8 points) Show that $\sigma_n(P) = 0$,
that is show that the smallest singular value of P is 0.

(c) (8 points) Now suppose in particular that P is the orthogonal projector onto $\text{span}\{v\}^\perp$ for a given nonzero vector $v \in \mathbb{C}^n$. Give an expression for P in terms of v.

(b) The unique projector onto $\mathbb{C}^n$ is I_n.

Since $P \neq I_n$, P is a projector onto a proper subspace of $\mathbb{C}^n$, that is $\text{Col}(P) \subset \mathbb{C}^n$.

Furthermore,

$$\text{Col}(P) \oplus \text{Null}(P) = \mathbb{C}^n,$$

~~As~~ $\text{Null}(P) = \{0\}$ is not possible, as this implies $\text{Col}(P) = \mathbb{C}^n$.

Hence,

$$Pv = 0 \quad \exists v \neq 0$$

$$\Longrightarrow \text{Rank}(P) \leq n-1$$

$$\Longrightarrow \sigma_n(P) = 0.$$

(c) The orthogonal projector onto $\text{span}\{v\}$ is given by

$$\tilde{P} = \frac{vv^T}{v^Tv}$$

The orthogonal projector onto $\text{span}\{v\}^\perp$ is the complementary one, that is

$$P = I - \tilde{P} = I - \frac{vv^T}{v^Tv}$$

Problem 4. (20 points) A matrix $H \in \mathbb{C}^{n \times n}$ is called Hessenberg if $h_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ such that $i - j > 1$. For instance, the matrix

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ -4 & 1 & 2 & 1 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 2 & -5 \end{bmatrix}$$

is Hessenberg.

For every matrix $A \in \mathbb{C}^{n \times n}$, there exists a unitary matrix $Q \in \mathbb{C}^{n \times n}$ and a Hessenberg matrix $H \in \mathbb{C}^{n \times n}$ such that

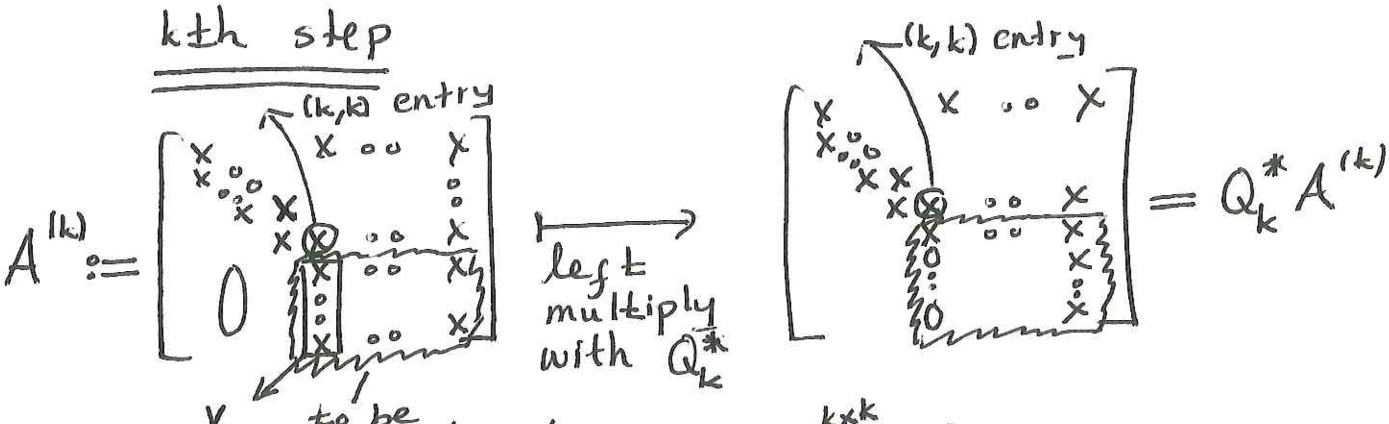
$$Q^* A Q = H. \tag{3}$$

Given a matrix $A \in \mathbb{C}^{n \times n}$, design an efficient algorithm, in particular write down a pseudocode, that computes a Hessenberg matrix $H \in \mathbb{C}^{n \times n}$ satisfying (3) for some unitary matrix $Q \in \mathbb{C}^{n \times n}$. Your algorithm does not need to form the unitary matrix Q explicitly, rather it suffices if it only returns H. Count the number of flops required by your algorithm.

We proceed column by column from 1st column to the ~~nth~~ $(n-2)$th column. (Nothing to do on columns ~~n-1, n~~ $n-1, n$)

At the kth step we introduce 0s below ~~(k,k)~~ $(k+1,k)$ entry by Householder reflectors.

<u>kth step</u>

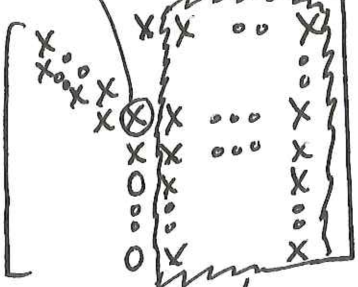


to be modified $A^{(k)}(k+1:n, k:n)$

$$Q_k^* = Q_k = \begin{bmatrix} I_k & 0 \\ 0 & H_k \end{bmatrix} \quad k \times k, \ (n-k) \times (n-k)$$

where $H_k := I - 2 q_k q_k^*$ with $q_k := \dfrac{v_k - \|v_k\|_2 e_1}{\| v_k - \|v_k\|_2 e_1 \|_2}$

(k,k) entry

$\underbrace{Q_k^* A^{(k)}}_{\tilde{A}^{(k)}} =$  $\xmapsto{\text{Right multiply with } Q_k}$ $Q_k^* A Q_k =: A^{(k+1)}$

only this portion will be affected by right multiplication with Q_k — $\tilde{A}^{(k)}(:, k+1:n)$

Pseudocode

for $k = 1, 2, \ldots, n-2$

$v \leftarrow A(k+1:n, k)$

$q \leftarrow \{v - \|v\|_2 e_1\} / \|v - \|v\|_2 e_1\|_2$

% Left multiply

① $A(k+1:n, k:n) \leftarrow A(k+1:n, k:n) - 2q\,(q^* A(k+1:n, k:n))$

% Right multiply

② $A(:, k+1:n) \leftarrow A(:, k+1:n) - \{A(:, k+1:n)(2q)\}\, q^*$

end

$H \leftarrow A$

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① requires $\sim 4(n-k)^2$ flops ② requires $\sim 4n(n-k)$ flops

$$\#\text{ of flops} \sim \sum_{k=1}^{n-2} \left[4(n-k)^2 + 4n(n-k)\right]$$

$$\sim \sum_{k=1}^{n} 4k^2 + 4nk$$

$$\simeq 4\,\frac{n(n+1)(2n+1)}{6} + 4n\,\frac{n(n+1)}{2} \sim \frac{10n^3}{3}$$