WI - PI

M 504
FALL 2018

Inner Product

$x, y \in \mathbb{R}^n$

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n = x^T y$$

$$\left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \end{bmatrix} \right\rangle = (1)(-3) + (2)(3)$$

$x, y \in \mathbb{C}^n$

$$\langle x, y\rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n = x^* y$$

$$\left\langle \begin{bmatrix} 1+2i \\ 3i \end{bmatrix}, \begin{bmatrix} 3 \\ 2-5i \end{bmatrix} \right\rangle = \begin{bmatrix} 1-2i & -3i \end{bmatrix} \begin{bmatrix} 3 \\ 2-5i \end{bmatrix}$$

$$= (1-2i)(3) + (-3i)(2-5i)$$

$$= 3 - 6i - 6i - 15 = -12 - 12i$$

Defn

An inner product on a real (complex) vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ $(\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C})$ such that

(i) $\langle v, v\rangle$ is real and positive $\forall v \in V, v \neq 0$,

(ii) $\langle v, w\rangle = \langle w, v\rangle$ $(\langle v, w\rangle = \overline{\langle w, v\rangle})$ for all $v, w \in V$,

(iii) $\langle v, \alpha w \rangle = \alpha \langle v, w \rangle \quad \forall v, w \in V, \ \forall \alpha \in \mathbb{R}$

$(\forall v, w \in V, \ \forall \alpha \in \mathbb{C})$

(iv) $\langle v, u+w \rangle = \langle v, u \rangle + \langle v, w \rangle \quad \forall v, w, u \in V$

Standard Inner Products

$x, y \in \mathbb{R}^n$ — $\langle x, y \rangle = x^T y$

$x, y \in \mathbb{C}^n$ — $\langle x, y \rangle = x^* y$

$A, B \in \mathbb{R}^{m \times n}$ — $\langle A, B \rangle = \operatorname{Trace}(A^T B)$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$

$A, B \in \mathbb{C}^{m \times n}$ — $\langle A, B \rangle = \operatorname{Trace}(A^* B)$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{a}_{ij} b_{ij}$$

$p, q \in \mathbb{P}_n$ — $\langle p, q \rangle = \int_a^b p(t) q(t) \, dt$

$a, b \in \mathbb{R}$ prescribed

Norms induced by inner products (Euclidean norms)

Let $v \in V$, where V is a vector space with an inner product $\langle \cdot, \cdot \rangle$.

The associated (Euclidean) norm is defined by

$$\|v\| := \sqrt{\langle v, v \rangle}$$

Standard Euclidean norms

$$x \in \mathbb{R}^n - \quad \|x\| = \sqrt{x^T x}$$
$$= \sqrt{x_1^2 + \cdots + x_n^2}$$

$$x \in \mathbb{C}^n - \quad \|x\| = \sqrt{x^* x}$$
$$= \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

$$A \in \mathbb{R}^{m \times n} - \quad \|A\| = \sqrt{\text{Trace}(A^T A)}$$
$$= \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$$

$$A \in \mathbb{C}^{m \times n} - \quad \|A\| = \sqrt{\text{Trace}(A^* A)}$$
$$= \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

$$p \in \mathbb{P}_n - \quad \|p\| = \sqrt{\int_a^b p^2(t)\, dt}$$

Norms in general

Defn

A norm on a vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying

(i) $\|v\| > 0 \quad \forall v \in V,\ v \neq 0$

(ii) $\|\alpha v\| = |\alpha|\,\|v\| \quad \forall v \in V,\ \forall \alpha \in \mathbb{R}$
$(\forall v \in V,\ \forall \alpha \in \mathbb{C})$

(iii) $\|v + w\| \leq \|v\| + \|w\|$

Common norms in $\mathbb{C}^n$ or $\mathbb{R}^n$

1-norm $\quad \|x\|_1 := |x_1| + \cdots + |x_n|$

∞-norm $\quad \|x\|_\infty := \max_{j=1,\dots,n} |x_j|$

p-norm $\quad \|x\|_p := \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$

Euclidean norm is referred as the 2-norm

$$\|x\|_2 = \sqrt[2]{|x_1|^2 + \cdots + |x_n|^2}$$

Common norms in $\mathbb{P}_n$

$$\|p\|_1 := \int_a^b |p(t)|\,dt$$

$$\|p\|_\infty := \max_{t\in[a,b]} |p(t)|$$

$$\|p\|_p := \sqrt[p]{\int_a^b |p(t)|^p\,dt}$$

All norms are equivalent in $\mathbb{C}^n$ or $\mathbb{R}^n$ (actually in any finite dimensional vector space) up to a scaling

Proposition

Let $\|\cdot\|_A$, $\|\cdot\|_B$ be two norms in $\mathbb{C}^n$ or $\mathbb{R}^n$.
There exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1\|v\|_A \leq \|v\|_B \leq c_2\|v\|_A \quad \forall v.$$

Ex

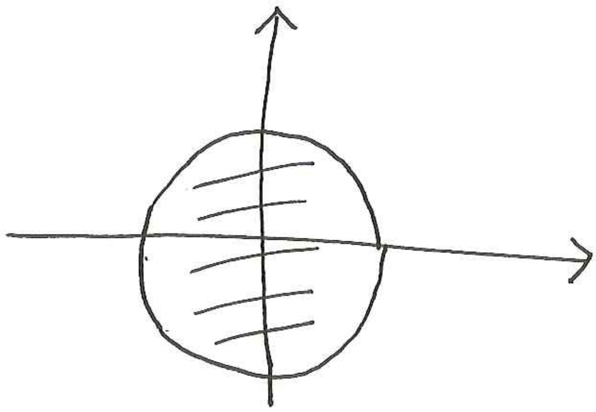
$$\|x\|_\infty = \max_{j=1,\dots,n} |x_j|$$

$$\leq |x_1| + \dots + |x_n| = \|x\|_1$$

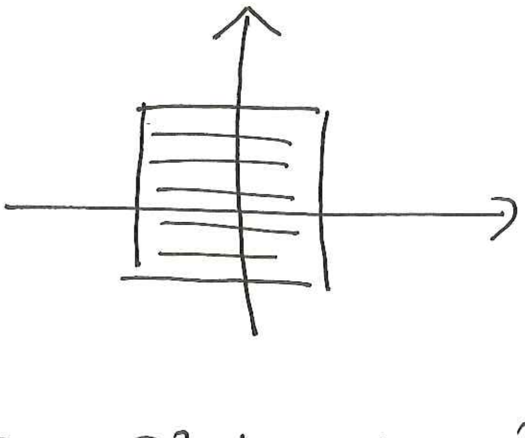
$$\leq n \cdot \max_{j=1,\dots,n} |x_j| = n \|x\|_\infty$$

But norms do affect the geometry

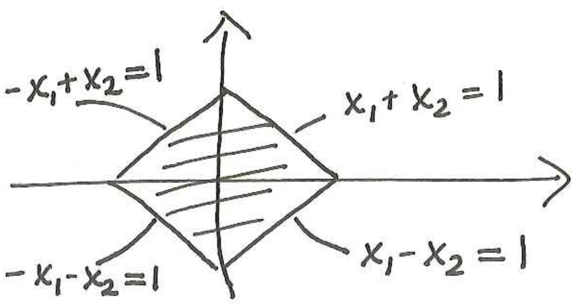
$$\{x \in \mathbb{R}^2 \mid \|x\|_2 \leq 1\}$$



$$\{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\}$$



$$\{x \in \mathbb{R}^2 \mid \|x\|_1 \leq 1\}$$



Orthogonality

V a vector space with an inner product $\langle \cdot, \cdot \rangle$

$u, v \in V$ are said to be orthogonal if

$$\langle u, v \rangle = 0$$

This is denoted by $u \perp v$.

Ex

* $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \perp \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = u_2$

as $\langle u_1, u_2 \rangle = u_1^T u_2 = 0.$

* $$A = \underbrace{\left(\frac{A + A^*}{2}\right)}_{A_H} + \underbrace{\left(\frac{A - A^*}{2}\right)}_{A_{SH}}$$

where $A_H^* = A_H$ (A_H is Hermitian)

$A_{SH}^* = -A_{SH}$ (A_{SH} is skew-Hermitian)

$$\langle A_H, A_{SH} \rangle = \frac{1}{4}\langle A, A \rangle + \cancel{\frac{1}{4}\langle A^*, A \rangle} - \cancel{\frac{1}{4}\langle A, A^* \rangle} - \frac{1}{4}\langle A^*, A^* \rangle$$

$$= \frac{1}{4}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} |\bar{a}_{ji}|^2 \right)$$

$$= \frac{1}{4}\left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 - \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ji}|^2 \right) = 0$$

$$A_H \perp A_{SH}$$

* $p(x) = x^2 - \frac{1}{2}$ $\left(\mathbb{P}_2 \text{ with, } \langle r, q \rangle := \int_{-1}^{1} r(x) q(x)\, dx \right.$

$$p \perp 1$$

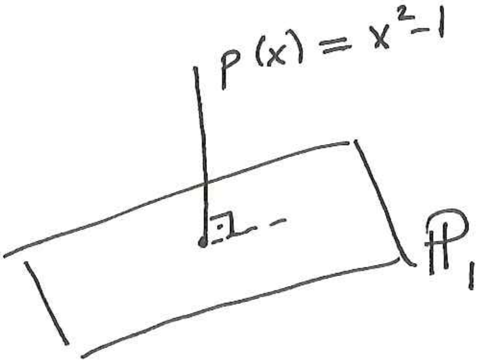
as $$\langle x^2 - \tfrac{1}{2}, 1 \rangle = \int_{-1}^{1} x^2 - \frac{1}{2}\, dx = \frac{x^3}{3} - \frac{x}{2} \Big|_{-1}^{1} = 0$$

$$p \perp x$$

as $$\langle x^2 - \tfrac{1}{2}, x \rangle = \int_{-1}^{1} x^3 - \frac{x}{2}\, dx = \frac{x^4}{4} - \frac{x^2}{4} \Big|_{-1}^{1} = 0$$

Consequently,

$$p \perp q \qquad \forall q \in \operatorname{span}\{1, x\} = \mathbb{P}_1$$



$p(x) = x^2 - 1$ - Legendre polynomial of degree 2.

Let $\|\cdot\|$ denote the Euclidean-norm associated with $\langle \cdot, \cdot \rangle$.

If $u \perp v$, then

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$
$$= \langle u,u \rangle + \cancel{\langle u,v \rangle} + \cancel{\langle v,u \rangle} + \langle v,v \rangle$$
$$= \|u\|^2 + \|v\|^2$$

<u>THM</u> (Pythagorean)
Let $u, v \in V$ be orthogonal.

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Now consider $p(t) := \|u - tv\|^2$
(u, v not necessarily orthogonal)

$$\langle u - tv, u - tv \rangle = \langle u,u \rangle + |t|^2 \langle v,v \rangle$$
$$- \bar{t} \langle v,u \rangle - t \langle u,v \rangle \geqslant 0$$

Setting $t = \overline{\langle u,v \rangle} / \|v\|^2$

$$\langle u,u \rangle + |\langle u,v \rangle|^2 / \|v\|^2 - \frac{|\langle u,v \rangle|^2}{\|v\|^2} - \frac{|\langle u,v \rangle|^2}{\|v\|^2} \geqslant 0$$

$$\Longrightarrow \|u\|^2 - \frac{|\langle u,v \rangle|^2}{\|v\|^2} \geqslant 0 \Longrightarrow |\langle u,v \rangle| \leqslant \|u\| \, \|v\|$$

<u>THM</u> (Cauchy-Schwarz Inequality)

$$|\langle u,v \rangle| \leqslant \|u\| \, \|v\| \quad \forall u, v \in V.$$

Defn (Orthonormal Set)

A set $\{u_1, \ldots, u_q\}$ in V is said to be orthonormal if

(i) $u_j \perp u_k \quad \forall j,k$ s.t $j \neq k$,

(ii) $\|u_j\| = 1 \quad \forall j$.

$$\left\{ \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$ is orthonormal

Defn (Orthonormal Basis)

A set $\{u_1, \ldots, u_q\}$ in V is an orthonormal basis for V if

(i) $\{u_1, \ldots, u_q\}$ is orthonormal,

(ii) $\text{span}\{u_1, \ldots, u_q\} = V$.

$$\left\{ \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is an orthonormal basis for $\mathbb{R}^3$.

Defn (Unitary Matrix)

A matrix $U \in \mathbb{C}^{n\times n}$ is said to be unitary if $U^*U = I_n$.

$$U^*U = I_n \iff \begin{aligned} [U^*U]_{jk} &= u_j^* u_k = 0 \quad \forall j,k \; j\neq k \\ [U^*U]_{jj} &= u_j^* u_j = 1 \quad \forall j \end{aligned}$$

$\iff \{u_1, \ldots, u_n\}$ is an orthonormal basis for $\mathbb{C}^n$.

$$U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \quad \text{is unitary.}$$

The following are all equivalent:

(i) U is unitary (i.e., $U^*U = I_n$).

(ii) $UU^* = I_n$.

(iii) Columns of U form an orthonormal basis for $\mathbb{C}^n$.

Gram-Schmidt Procedure

To generate an orthonormal basis for a subspace $S = \text{span}\{v_1, \ldots, v_q\}$ (also shows the existence of an orthonormal basis)

$S_1 = \text{span}\{v_1\}$

$\{u_1\}$, $u_1 = v_1 / \|v_1\|$

is an orthonormal basis

$S_2 = \text{span}\{v_1, v_2\}$

$\{u_1, u_2\}$ an orthonormal basis

$v_2 = \alpha_1 u_1 + \alpha_2 u_2$

$\Longrightarrow \langle u_1, v_2 \rangle = \alpha_1 + \cancel{\alpha_2 \langle u_1, u_2 \rangle}$

Hence,

$u_2 = \tilde{u}_2 / \|\tilde{u}_2\|$ where $\tilde{u}_2 = v_2 - \langle u_1, v_2 \rangle u_1$

$S_j = \text{span}\{v_1, \dots, v_j\}$

$\{u_1, \dots, u_j\}$ an orthonormal basis

$$v_j = \alpha_1 u_1 + \dots + \alpha_{j-1} u_{j-1} + \alpha_j u_j$$

$$\Longrightarrow \quad \langle u_k, v_j \rangle = \alpha_k \qquad k = 1, \dots, j-1$$

Hence,

$$u_j = \tilde{u}_j / \|\tilde{u}_j\| \quad \text{where } \tilde{u}_j := v_j - \sum_{k=1}^{j-1} \langle u_k, v_j \rangle u_k$$

Ex

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$u_1 = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}}, \qquad \tilde{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left\langle \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

THM (Gram-Schmidt Procedure)

Let $S = \text{span}\{v_1, \dots, v_q\}$. The set $\{u_1, \dots, u_q\}$ defined by

(i) $u_1 = v_1 / \|v_1\|$

(ii) $u_j = \tilde{u}_j / \|\tilde{u}_j\|, \quad \tilde{u}_j = v_j - \sum_{k=1}^{j-1} \langle u_k, v_j \rangle u_k$

$j = 2, \dots, q$

is an orthonormal basis for S.

Implementation in $\mathbb{C}^n$

Algorithm (Gram – Schmidt)

for $j = 1, \ldots, q$

$u_j \leftarrow v_j$

  for $k = 1, \ldots, j-1$

    $u_j \leftarrow u_j - (u_k^* v_j)\, u_k$

  end

$u_j \leftarrow u_j / \|u_j\|$

end

Flop count

Count of # of arithmetic operations

* $u_k^* v_j$ $2n-1$ flops
* $(u_k^* v_j)\, u_k$ n flops
* $u_j - (u_k^* v_j)\, u_k$ n flops
* $u_j / \|u_j\|$ $3n-1$ flops

$$\text{Total \# flops} = \sum_{j=1}^{q} \left\{ \sum_{k=1}^{j-1} 4n-1 \right\} + (3n-1)$$

$$\sim 4 \sum_{j=1}^{q} \sum_{k=1}^{j} n = 4 \sum_{j=1}^{q} nj$$

$$\sim 2nq^2$$

Asymptotic notations

$f(n) \sim g(n)$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

$f(n) = O(g(n))$ means $\exists c$ s.t.
$|f(n)| \leqslant c|g(n)|$
for all large n.

$f(n) = o(g(n))$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

$3n^2 = O(n^2)$, $\quad 3n^2 \neq o(n^2)$

$3n^2 = O(n^3)$, $\quad 3n^2 = o(n^3)$

Eigenvalues

A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{C}^{n\times n}$ if

$$(*) \quad Av = \lambda v \quad \exists v \in \mathbb{C}^n,\ v \neq 0$$

A vector $v \neq 0$ satisfying $(*)$ is called an eigenvector corresponding to λ.

$$\underset{\exists v \neq 0}{Av = \lambda v} \iff (A - \lambda I_n)v = 0 \quad \exists v \neq 0$$

$$\iff \underbrace{\det(A - \lambda I_n)}_{\substack{p(\lambda) := \\ \text{characteristic} \\ \text{polynomial of } A.}} = 0$$

Ex

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad \det\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$$

eigenvalues

$\lambda_1 = 4 \quad \lambda_2 = 2$

$$(A - 4I)v_1 = 0 \qquad (A - 2I)v_2 = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \underset{\substack{\swarrow \\ \text{eigenvector} \\ \text{corr. to } \lambda_1}}{v_1} = 0 \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underset{\substack{\swarrow \\ \text{eigenvector} \\ \text{corr. to } \lambda_2}}{v_2} = 0$$

Schur Factorization

THM

Every matrix $A \in \mathbb{C}^{n\times n}$ has a factorization of the form

$$A = Q T Q^*$$

where $Q \in \mathbb{C}^{n\times n}$ is unitary,

$T \in \mathbb{C}^{n\times n}$ is upper triangular.

Proof

By induction on n.

The base case $n=1$ is trivial. (i.e. $A = 1\cdot A\cdot 1^*$)

As for the inductive case, let λ be an eigenvalue, $v \in \mathbb{C}^n$ be a corresponding unit eigenvector, and $\tilde{Q} = [v \ \underbrace{q_2 \cdots q_n}_{\tilde{\tilde{Q}}}]$ be unitary.

$$\tilde{Q}^* A \tilde{Q} = \tilde{Q}^*[\lambda v \ \ A\tilde{\tilde{Q}}] = [\lambda \tilde{Q}^* v \ \ \tilde{Q}^* A \tilde{\tilde{Q}}]$$

$$= \begin{bmatrix} \lambda & \overset{1\times(n-1)}{w} \\ \underset{(n-1)\times 1}{0} & \underset{(n-1)\times(n-1)}{\tilde{A}} \end{bmatrix}$$

By the inductive hypothesis, $\tilde{A} = \hat{Q}\hat{T}\hat{Q}^*$ where $\hat{T} \in \mathbb{C}^{(n-1)\times(n-1)}$ is upper triangular, $\hat{Q} \in \mathbb{C}^{(n-1)\times(n-1)}$ is unitary, so

$$\tilde{Q}^* A \tilde{Q} = \begin{bmatrix} \lambda & w \\ 0 & \hat{Q}\hat{T}\hat{Q}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \hat{Q} \end{bmatrix} \begin{bmatrix} \lambda & w\hat{Q} \\ 0 & \hat{T} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}^* \end{bmatrix}$$

$$\Longrightarrow A = \underbrace{\tilde{Q}\begin{bmatrix} 1 & 0 \\ 0 & \hat{Q} \end{bmatrix}}_{\text{unitary} \leftarrow Q} \underbrace{\begin{bmatrix} \lambda & w\hat{Q} \\ 0 & \hat{T} \end{bmatrix}}_{\text{upper trian.} \leftarrow T} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}^* \end{bmatrix} \tilde{Q}^*}_{Q^*}.$$

$\square$

Ex

$$\begin{bmatrix} 6 & 2 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Hermitian matrices

$A \in \mathbb{C}^{n \times n}$, $A^* = A$ with the

Schur factorization

$$A = QTQ^*.$$

$$QTQ^* = (QTQ^*)^* \iff QTQ^* = QT^*Q^*$$

$$\iff T = T^*$$

$\iff$ T is diagonal with real entries along diagonal.

Hence, A has a factorization of the form

$$A = [q_1 \cdots q_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^* \\ \vdots \\ q_n^* \end{bmatrix}$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$

$$\iff A[q_1 \cdots q_n] = [q_1 \cdots q_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\iff Aq_j = \lambda_j q_j \quad j = 1, \ldots, n$$

A Hermitian matrix A has

* real eigenvalues $\lambda_1, \dots, \lambda_n$,
* the corresponding eigenvectors $q_1, \dots, q_n$ are such that $\{q_1, \dots, q_n\}$ is orthonormal.

Hermitian positive semi-definite matrices

$A \in \mathbb{C}^{n \times n}$, $A^* = A$ and

$$v^* A v \geq 0 \qquad \forall v \in \mathbb{C}^n.$$

<u>THM</u>

Let $A \in \mathbb{C}^{n \times n}$ be Hermitian. The following are equivalent:

(i) A is positive semi-definite.

(ii) all eigenvalues of A are nonnegative.

<u>Proof</u>

$\sim$(ii) $\Longrightarrow$ $\sim$(i)

Suppose A has a negative eigenvalue λ and v be a corresponding unit eigenvector.

$$v^* A v = v^* \lambda v = \lambda \|v\|^2 = \lambda < 0$$

Hence, A is not positive semi-definite

(ii) $\Rightarrow$ (i)

Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of A all nonnegative, $v_1, \dots, v_n$ be the corresponding eigenvectors such that $\{v_1, \dots, v_n\}$ is an orthonormal basis.

Every $v \in \mathbb{C}^n$ can be written as

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, and

$$\begin{aligned} v^* A v &= (\bar{\alpha}_1 v_1^* + \dots + \bar{\alpha}_n v_n^*)(\alpha_1 v_1 + \dots + \alpha_n v_n) \\ &= |\alpha_1|^2 v_1^* v_1 + \dots + |\alpha_n|^2 v_n^* v_n \\ &= |\alpha_1|^2 + \dots + |\alpha_n|^2 \geq 0. \end{aligned}$$

Hence, A is positive semi-definite.