W2 - P1 | M 504 F 18

Singular value decomposition (SVD)

$A \in \mathbb{C}^{m \times n}$ (for simplicity assume $m \geq n$)

A^*A is Hermitian positive semi-definite

H (i) $(A^*A)^* = A^*(A^*)^* = A^*A$

PSD (ii) $\forall v \in \mathbb{C}^n$, $v^* A^* \underbrace{Av}_{y :=} = y^* y$

$= |y_1|^2 + \ldots + |y_m|^2 \geq 0$

Define the singular values of A

$\sigma_1 \geq \ldots \geq \sigma_n$

as the square-roots of eigenvalues of A^*A.

Define $v_1, \ldots, v_n$ such that

$$A^*A v_j = \sigma_j^2 v_j \quad j = 1, \ldots, n$$

and $\{v_1, \ldots, v_n\}$ is orthonormal.

v_j - right singular vector corresponding to σ_j

Suppose $\sigma_1, \dots, \sigma_r > 0$, but $\sigma_{r+1} = \dots = \sigma_n = 0$.
Define $u_1, \dots, u_r$ by

$$A v_j = \sigma_j u_j \qquad j = 1, \dots, r.$$

and $u_{r+1}, \dots, u_n \in \mathrm{Col}(A)^\perp$ s.t.
$\{u_{r+1}, \dots, u_n\}$ is orthonormal.

Proposition

$\{u_1, u_2, \dots, u_n\}$ is orthonormal.

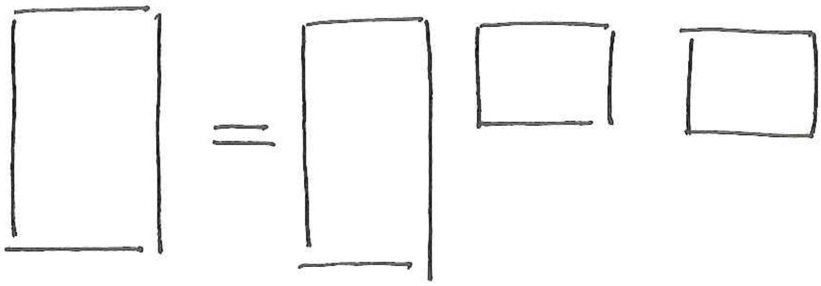
u_j – left singular vector
corresponding to σ_j

$$A \underbrace{[v_1 \ v_2 \ \dots \ v_n]}_{\hat{V} \in \mathbb{C}^{n \times n}} = [\sigma_1 u_1 \ \dots \ \sigma_n u_n]$$

$$= \underbrace{[u_1 \ u_2 \ \dots \ u_n]}_{\hat{U} \in \mathbb{C}^{m \times n}} \underbrace{\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix}}_{\substack{\hat{\Sigma} \in \mathbb{R}^{n \times n} \\ \sigma_1, \dots, \sigma_n \geq 0}}$$

Reduced SVD

$$A = \underset{m \times n}{\hat{U}} \; \underset{n \times n}{\hat{\Sigma}} \; \underset{n \times n}{\hat{V}^*}$$



$\hat{U}$ - has orthonormal columns

$\hat{\Sigma}$ - diagonal, with real nonnegative entries

$\hat{V}$ - unitary

Ex

$$A = \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}$$

$$\begin{aligned} \det(A^*A - \lambda I) &= (10-\lambda)^2 - 36 \\ &= \lambda^2 - 20\lambda + 64 \\ &= (\lambda - 16)(\lambda - 4) \end{aligned}$$

$\sigma_1 = 4, \quad \sigma_2 = 2.$

Right singular vectors

$$(A^*A - 16I)\, v_1 = 0$$

$$\begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix} v_1 = 0 \implies v_1 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

e.g.

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A^*A - 4I)\, v_2 = 0$$

$$\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} v_2 = 0 \implies v_2 = c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

e.g.

$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Left singular vectors

$$u_1 = \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} v_1 \Big/ \sigma_1 = \frac{1}{4\sqrt{2}} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} v_2 \Big/ \sigma_2 = \frac{1}{2\sqrt{2}} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Reduced SVD

$$\begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

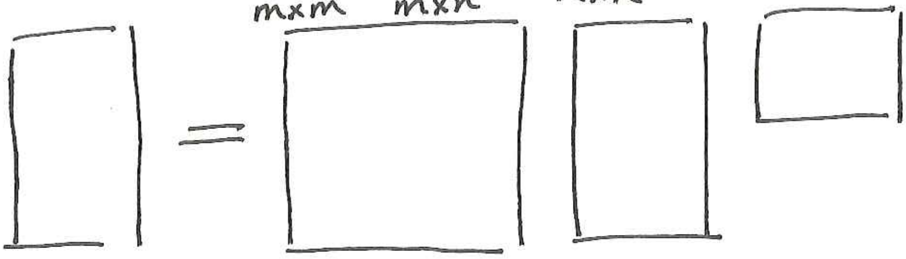
$$A = \hat{U}\hat{\Sigma}\hat{V}^*$$

$$= \underbrace{\begin{bmatrix} \hat{U} & \tilde{U} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \hat{\Sigma} \\ 0 \end{bmatrix}}_{\Sigma} \hat{V}^*$$

(Augment $m \times n$ $\hat{U}$ with $\tilde{U}$ so that $[\hat{U}\ \tilde{U}]$ is $n \times n$ unitary

Full SVD

$$A = \underset{m \times m}{U} \ \underset{m \times n}{\Sigma} \ \underset{n \times n}{V^*} \quad (V = \hat{V})$$



U, V – unitary

Σ – diagonal, with nonnegative real entries

Ex

Reduced SVD

$$\begin{bmatrix} 10 & -5 \\ -2 & -14 \\ 10 & -5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 10\sqrt{3} & 0 \\ 0 & 5\sqrt{6} \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

Full SVD

$$\begin{bmatrix} 10 & -5 \\ -2 & -14 \\ 10 & -5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{3} & 0 \\ 0 & 5\sqrt{6} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

Remark

$$A^* A v_j = \sigma_j^2 v_j \qquad \left(\text{For simplicity assume } \sigma_j \neq 0 \right.$$

$$A^*(\sigma_j u_j) = \sigma_j^2 v_j$$

$$A^* u_j = \sigma_j v_j$$

Hence, a triple of (σ_j, u_j, v_j) satisfies

$$A v_j = \sigma_j u_j \quad \text{and} \quad A^* u_j = \sigma_j v_j$$

(Above equations hold even if $\sigma_j = 0$.)

Applications of SVD

Matrix 2-norm

$A \in \mathbb{C}^{m \times n}$

$$\|A\|_2 = \max_{\substack{v \in \mathbb{C}^n \\ \|v\|_2 = 1}} \|Av\|_2$$

For all $v \in \mathbb{C}^n$ s.t $\|v\|_2 = 1$

$$\|Av\|_2 = \|A(\alpha_1 v_1 + \cdots + \alpha_n v_n)\|_2 \quad \exists \alpha_1, \ldots, \alpha_n$$

$$= \|\alpha_1 \sigma_1 v_1 + \cdots + \alpha_n \sigma_n v_n\|_2$$

$$= \sqrt{\sigma_1^2 |\alpha_1|^2 + \cdots + \sigma_n^2 |\alpha_n|^2}$$

Note also $|\alpha_1|^2 + \cdots + |\alpha_n|^2 = 1$, since

$$1 = \|\alpha_1 v_1 + \cdots + \alpha_n v_n\|_2^2$$

$$= (\alpha_1 v_1 + \cdots + \alpha_n v_n)^* (\alpha_1 v_1 + \cdots + \alpha_n v_n)$$

$$= |\alpha_1|^2 + \cdots + |\alpha_n|^2$$

Hence, for all $v \in \mathbb{C}^n$ s.t. $\|v\|_2 = 1$

$$(+) \quad \|Av\|_2 \leqslant \sqrt{\sigma_1^2 \underbrace{(|\alpha_1|^2 + \cdots + |\alpha_n|^2)}_{1}} = \sigma_1$$

Additionally

$$(++) \quad \|Av_1\|_2 = \|\sigma_1 u_1\|_2 = \sigma_1 .$$

(+) and (++) combined imply

$$\|A\|_2 = \sigma_1$$

Ex

$$A = \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} \qquad \|A\|_2 = \sigma_1 = 4$$

Column space of A

The range of

$$f : \mathbb{C}^n \to \mathbb{C}^m, \quad f(v) = Av$$

$$\begin{aligned} \mathrm{Col}(A) &= \{ Av \mid v \in \mathbb{C}^n \} \\ &= \mathrm{span}\{a_1, \ldots, a_n\} \end{aligned}$$

Suppose $\sigma_1, \ldots, \sigma_r > 0$,

$$\sigma_{r+1} = \cdots = \sigma_n = 0.$$

Then

$$\begin{aligned} \mathrm{Col}(A) &= \{ A(\alpha_1 v_1 + \cdots + \alpha_n v_n) \mid \alpha_1, \ldots, \alpha_n \in \mathbb{C} \} \\ &= \{ \alpha_1 \sigma_1 u_1 + \cdots + \alpha_r \sigma_r u_r \mid \alpha_1, \ldots, \alpha_r \in \mathbb{C} \} \\ &= \{ \beta_1 u_1 + \cdots + \beta_r u_r \mid \beta_1, \ldots, \beta_r \in \mathbb{C} \} \\ &= \mathrm{span}\{u_1, \ldots, u_r\} \end{aligned}$$

$\{u_1, \ldots, u_r\}$ an orthonormal basis for $Col(A)$.

Ex

$$\underbrace{\begin{bmatrix} 10 & -5 \\ -2 & 14 \\ 10 & -5 \end{bmatrix}}_{A} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 10\sqrt{3} & 0 \\ 0 & 5\sqrt{6} \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$ is an orthonormal basis for $Col(A)$.

$$\text{Rank}(A) = \dim Col(A)$$
$$= r \quad (\#\text{ nonzero singular values})$$

Low Rank Approximation

$A \in \mathbb{C}^{m \times n}$, $p \in \mathbb{Z}^+$

Find $B \in \mathbb{C}^{m \times n}$ of rank $\leq p$ s.t.

$$\|B - A\|_2$$

is as small as possible.

$\exists B \in \mathbb{C}^{m \times n}$ s.t.

(i) $\text{Rank}(B) \leqslant p$

(ii) $\|B - A\|_2 = \sigma_{p+1}$

Namely

$$B = \underbrace{[u_1 \cdots u_n]}_{\hat{U}} \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_p & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \underbrace{\begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix}}_{\hat{V}^*}$$

Ex

See Ex on page 3, for that example set

$$B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 10\sqrt{3} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$= 10\sqrt{3} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ 6 & -8 \\ 6 & -8 \end{bmatrix}$$

$\|B - A\|_2 = \sigma_2 = 5\sqrt{6}$ $\qquad$ $\text{Rank}(B) = 1$

THM

For all $B \in \mathbb{C}^{m \times n}$ s.t. $\text{Rank}(B) \leqslant p$, we have

$$\|B - A\|_2 \geqslant \sigma_{p+1}.$$

Proof

The dimension of the null space of B

$$\text{Null}(B) := \{ v \in \mathbb{C}^n \mid Av = 0 \}$$

is at least $n-p$.

But then, letting $S := \text{span}\{v_1, \ldots, v_p, v_{p+1}\}$,

$$\text{Null}(B) \cap S \neq \{0\},$$

as $\dim \text{Null}(B) + \dim S \geq n+1$. Take any nonzero $\tilde{v} \in \text{Null}(B) \cap S$, assume (WLOG) $\|\tilde{v}\|_2 = 1$. It follows that

$$\|B-A\|_2 \geq \|(B-A)\tilde{v}\|_2$$
$$= \|A\tilde{v}\|_2 \geq \sigma_{p+1}. \quad \square$$

Corollary (Eckart-Young)

$$\min_{\substack{B \in \mathbb{C}^{m\times n} \\ \text{Rank}(B) \leq p}} \|B-A\|_2 = \sigma_{p+1}$$
$$= \|B_* - A\|_2$$

where

$$B_* = [u_1 \cdots u_n] \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_p & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} \begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix}.$$

Ex

① $$\underbrace{\begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix}}_{A} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Nearest rank 1 matrix

$$B_* = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

at a distance $\|B_* - A\|_2 = \sigma_2 = 2$.

(2) Nearest singular matrix

$A \in \mathbb{C}^{n \times n}$

$$\min \{ \|B - A\|_2 \mid B \in \mathbb{C}^{n \times n} \text{ s.t. } \det(B) = 0 \}$$

$$= \min \{ \|B - A\|_2 \mid B \in \mathbb{C}^{n \times n} \text{ s.t. } \operatorname{Rank}(B) \leq n-1 \}$$

$$= \sigma_n$$

$$B_* = \sigma_1 u_1 v_1^* + \cdots + \sigma_{n-1} u_{n-1} v_{n-1}^*$$

is a nearest singular matrix.