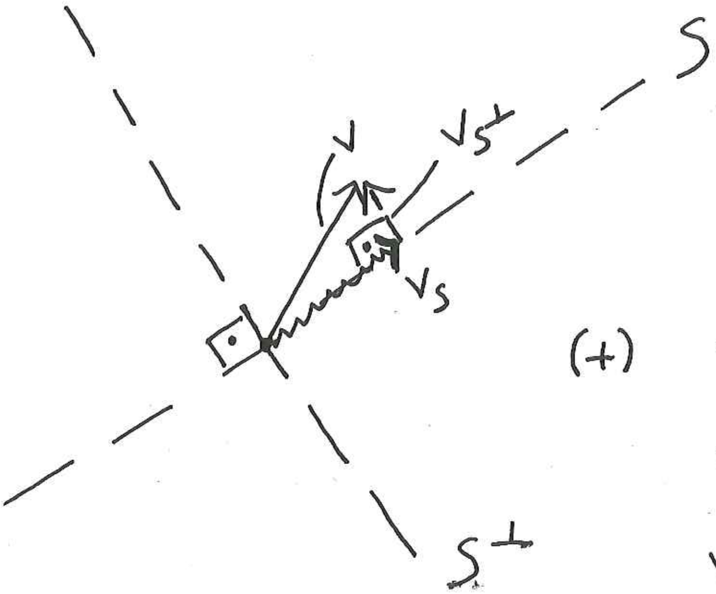
Orthogonal Projectors

S — subspace of $\mathbb{C}^n$



$$(+) \quad v = v_S + v_{S^\perp}$$

$$v_S \in S$$

$$v_{S^\perp} \in S^\perp$$

v_S - orthogonal projection of v onto S

Note that decomposition $(+)$ is unique, as

$$S \oplus S^\perp = \mathbb{C}^n$$

Orthogonal projector onto S

$P \in \mathbb{C}^{n \times n}$ such that

$$Pv = v_S \qquad \forall v \in \mathbb{C}^n.$$

Ex

$$S = \text{span}\left\{\begin{bmatrix}-3/5\\4/5\end{bmatrix}\right\}$$

$$v = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{v} = \underbrace{\alpha_1\begin{bmatrix}-3/5\\4/5\end{bmatrix}}_{v_S} + \underbrace{\alpha_2\begin{bmatrix}4/5\\3/5\end{bmatrix}}_{v_{S^\perp}}$$

$$\begin{bmatrix}-3/5 & 4/5\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \alpha_1\begin{bmatrix}-3/5 & 4/5\end{bmatrix}\begin{bmatrix}-3/5\\4/5\end{bmatrix}$$

$$\Longrightarrow \alpha_1 = -3/5$$

$$v_S = \begin{bmatrix}9/25\\-12/25\end{bmatrix} \quad \left(v - v_S = \begin{bmatrix}16/25\\12/25\end{bmatrix} \in S^\perp\right)$$

Orthogonal projector onto $\text{span}\left\{\begin{bmatrix}-3/5\\4/5\end{bmatrix}\right\}$

$$v = \underbrace{\alpha_1\begin{bmatrix}-3/5\\4/5\end{bmatrix}}_{v_S} + \alpha_2\begin{bmatrix}4/5\\3/5\end{bmatrix}$$

$$\begin{bmatrix}-3/5 & 4/5\end{bmatrix} v = \alpha_1\begin{bmatrix}-3/5 & 4/5\end{bmatrix}\begin{bmatrix}-3/5\\4/5\end{bmatrix}$$

$$\Longrightarrow \alpha_1 = \begin{bmatrix}-3/5 & 4/5\end{bmatrix} v$$

Hence,

$$v_S = \begin{bmatrix}-3/5\\4/5\end{bmatrix}\begin{bmatrix}-3/5 & 4/5\end{bmatrix} v = \begin{bmatrix}9/25 & -12/25\\-12/25 & 16/25\end{bmatrix} v$$

$$(++)\quad P = \begin{bmatrix}9/25 & -12/25\\-12/25 & 16/25\end{bmatrix}$$

Observations regarding (++)

$$P^2 = P \quad \text{and} \quad P^T = P$$

Suppose

$S = \text{span}\{a_1, \dots, a_q\}$,

$\{a_1, \dots, a_q\}$ is linearly independent.

Letting $A = [a_1 \cdots a_q] \in \mathbb{C}^{n \times q}$,

v any vector in $\mathbb{C}^n$.

$$v = \underbrace{A\alpha}_{v_S} + v_{S^\perp}$$

$$A^* v = A^* A \alpha + \underbrace{A^* v_{S^\perp}}_{0,\ \text{i.e.,}\ a_1, \dots, a_q \perp v_{S^\perp}}$$

$$\Longrightarrow \alpha = (A^* A)^{-1} A^* v$$

Hence,

$$v_S = A(A^* A)^{-1} A^* v$$

$$P = A(A^* A)^{-1} A^*$$

is the orthogonal projector onto S.

Example

$$S = \text{span}\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$$

$$A = \begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix}$$

$$P = \begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix} \Bigg(\underbrace{\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix} \begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix}}_{\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}} \Bigg)^{-1} \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix} \left(\frac{1}{3} \begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix} \right) \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix} \begin{bmatrix}2 & 1 & -1\\-1 & 1 & 2\end{bmatrix} = \frac{1}{3} \begin{bmatrix}2 & 1 & -1\\1 & 2 & 1\\-1 & 1 & 2\end{bmatrix}$$

Now suppose

$$S = \text{span}\{u_1, \dots, u_q\}$$

$\{u_1, \dots, u_q\}$ is orthonormal.

Letting $U = [u_1 \cdots u_q] \in \mathbb{C}^{n \times q}$,

$$P = U(\underbrace{U^* U}_{I_q \text{ - } q \times q \text{ identity matrix}})^{-1} U^*$$

$$= U U^*$$

Ex

$$S = \operatorname{span}\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

$$U = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{bmatrix}$$

THM

Let $P \in \mathbb{C}^{n \times n}$ be a projector. Then P is an orthogonal projector if and only if $P^* = P$.

Proof

Suppose P is an orthogonal projector onto $S = \operatorname{span}\{a_1, \ldots, a_q\}$, where $\{a_1, \ldots, a_q\}$ is linearly independent.

Setting $A = [a_1 \ldots a_q]$, we have

$$P = A(A^T A)^{-1} A^*$$

which satisfies $P^* = P$.

(IT IS NOT ESSENTIAL TO READ THIS)

Now suppose $P^* = P$. We have already established that P is a projector onto $\mathrm{Col}(P)$ along $\mathrm{Null}(P)$.

To conclude that P is an orthogonal projector, it suffices to show

$$\mathrm{Null}(P) = \mathrm{Col}(P)^\perp$$

To this end, let $z \in \mathrm{Null}(P)$. But then for every $y \in \mathrm{Col}(P)$,

$$\begin{aligned} y^* z &= (P\tilde{y})^* z \quad \exists\, \tilde{y} \in \mathbb{C}^n \\ &= \tilde{y}^* P^* z \\ &\overset{\text{as } P^* = P}{=} \tilde{y}^* P z \overset{\text{as } z \in \mathrm{Null}(P)}{=} 0, \end{aligned}$$

so $y \perp z$ implying $z \in \mathrm{Col}(P)^\perp$.

Conversely, let $z \in \mathrm{Col}(P)^\perp$. Then

~~for every~~ $\cancel{y \in \mathrm{Col}(P)}$,

$$\cancel{\begin{aligned} 0 &= y^* z \\ &= (P\tilde{y})^* z \quad \exists\, \tilde{y} \in \mathbb{C}^n \\ &= \tilde{y}^* P z \end{aligned}}$$

~~In particular~~

$$\begin{aligned} 0 &= (Pz)^* z \quad \left(\text{since } Pz \in \mathrm{Col}(P)\right) \\ &= z^* P z = z^* P^2 z = (Pz)^* Pz \end{aligned}$$

implying $Pz = 0$, that is $z \in \mathrm{Null}(P)$. □