Maximum Sets in a Finite Projective Space

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In a projective plane PG$(2, q)$ over the field $\mathbf{F}_q$ of q elements, a $(k; n)$-*arc* is a set of k points in the plane with at most n on any line and some line containing exactly n points of the set. These have been most studied for the case $n = 2$ and they correspond to an MDS code of dimension 3. The largest value of k for a $(k; n)$-arc is denoted $m_n(2, q)$.

More generally, a $(k; r, s; d, q)$-*set* K is defined to be a set satisfying the following properties:

(a) the set K consists of k points of PG(d, q) and is not contained in a proper subspace;

(b) some subspace Π_s contains r points of K, but no Π_s contains $r + 1$ points of K;

(c) there is a subspace Π_{s+1} containing $r + 2$ points of K.

So a $(k; n)$-arc is a $(k; n, 1; 2, q)$-set.

A $(k; r, s; d, q)$-set is *complete* if it is maximal with respect to inclusion; that is, it is not contained in a $(k + 1; r, s; d, q)$-set.

The main problems are the following.

(I) Find $m(r, s; d, q)$, the maximum value of k.

(II) Classify these sets of maximum size.

(III) Find $m'(r, s; d, q)$, the size k of the second largest complete $(k; r, s; d, q)$-set.

The progress of these problems in the last 25 years is considered, concentrating mainly on the two cases:

(i) $(k; n, 1; 2, q)$-sets;

(ii) $(k; d, d - 1; d, q)$-sets.