Appendix

1 Proof of Lemma 1

This lemma states that all operators preserve some of the four different properties, namely monotonicity, Upper-Bounded Differences (UBD), Lower-Bounded Differences (LBD), and concavity. We devote a subsection to each of these properties to prove the corresponding statements.

1.1 Monotonicity

This section will show that all the queueing operators defined in Section 3 preserve the monotonicity of any monotone function \( f(x) \). More explicitly, we prove the following inequality is true when \( T \) is one of the operators \( T_{ARR}, T_{SARR}, T_{DEP}, T_{CD}, T_{QPRC}, T_{BADM} \):

\[
Tf(x) \geq Tf(x+1),
\]

whenever \( f(x) \) is a non-increasing function of \( x \).

1.1.1 Monotonicity preserved by \( T_{ARR} \)

We can write inequality (1) for \( T_{ARR} \) as follows by using the definition of the operator:

\[
a(x)f(x+1) + [1-a(x)]f(x) \geq a(x+1)f(x+2) + [1-a(x+1)]f(x+1),
\]

which can be rearranged, since \( a(x+1) \leq a(x) \) for all \( x \), to obtain:

\[
[1-a(x)]f(x) + a(x+1)f(x+1) \geq [1-a(x)]f(x+1) + a(x+1)f(x+2).
\]

The inequality is valid for both lines by the monotonicity of \( f(x) \).

For systems with capacity of \( K \) (including the servers), we need to observe the boundary effects, i.e., the state \( x = K-1 \). Since the system can not admit new arrivals when there are already \( K \) customers in the system, \( a(K-1) > 0 \) and \( a(K) = 0 \). Therefore, the second terms of both LHS and RHS of inequality (2) becomes 0, so that inequality (1) still holds for \( T_{ARR} \).

1.1.2 Monotonicity preserved by \( T_{DEP} \)

The proof is similar to the previous one. As \( b(x) \leq b(x+1) \), we can write and rearrange inequality (1) as follows:

\[
b(x)f(x-1) + [1-b(x+1)]f(x) \geq b(x)f(x) + [1-b(x+1)]f(x+1),
\]

and the inequality is true by the monotonicity of \( f(x) \). Therefore, the function, \( T_{DEP}f(x) \), will be non-increasing in \( x \) if \( f(x) \) is non-increasing in \( x \). Since the capacity of the system does not affect the departures, we do not need to consider those systems separately.
1.1.3 Monotonicity preserved by $T_{CD}$

Let $\pi_x$ and $\pi_{x+1}$ be the optimal service rates in states $x$ and $x+1$, respectively. Then, we can write inequality (1) for $T_{CD}$ as follows:

$$-c_{\pi_x} + \pi_x f(x-1) + (1 - \pi_x) f(x) \geq -c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1 - \pi_{x+1}) f(x+1). \quad (3)$$

By the definition of the operator and the monotonicity of $f(x)$, we have:

$$-c_{\pi_x} + \pi_x f(x-1) + (1 - \pi_x) f(x) \geq -c_{\pi_{x+1}} + \pi_{x+1} f(x-1) + (1 - \pi_{x+1}) f(x), \text{ and}$$

$$-c_{\pi_{x+1}} + \pi_{x+1} f(x-1) + (1 - \pi_{x+1}) f(x) \geq -c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1 - \pi_{x+1}) f(x+1).$$

These two inequalities together imply that inequality (3) holds. Thus, $T_{CD} f(x)$ is non-increasing in $x$. As in $T_{DEP}$, we do not need to consider the capacitated systems.

1.1.4 Monotonicity preserved by $T_{QPRC}$

Let $p_x$ and $p_{x+1}$ be the optimal prices for the states $x$ and $x+1$, respectively. Then, inequality (1) for $T_{QPRC}$ is as follows:

$$\bar{F}(p_x)[f(x+1) + p_x] + F(p_x)f(x) \geq \bar{F}(p_{x+1})[f(x+2) + p_{x+1}] + F(p_{x+1})f(x+1). \quad (4)$$

As in the controlled departure operator, we have the following inequalities by the definition of the pricing operator and the monotonicity of $f(x)$:

$$\bar{F}(p_x)[f(x+1) + p_x] + F(p_x)f(x) \geq \bar{F}(p_{x+1})[f(x+1) + p_{x+1}] + F(p_{x+1})f(x)$$

$$\bar{F}(p_{x+1})[f(x+1) + p_{x+1}] + F(p_{x+1})f(x) \geq \bar{F}(p_{x+1})[f(x+2) + p_{x+1}] + F(p_{x+1})f(x+1).$$

Combining these inequalities ensures that inequality (4) holds, and so $T_{QPRC}$ is non-increasing in $x$. The pricing operator can also be used in the capacitated queues. In this case, we need to observe the boundary effects in state $x = K - 1$. However, since we will use the optimality of $p_{K-1}$, the foregoing proof is still true. Note that $p_K$ can be taken as a large enough price to set $\bar{F}(p_K) = 0$.

1.1.5 Monotonicity preserved by $T_{B_{ADM}}$

Let $\kappa^i_{xB_i}$ and $\kappa^i_{x+1B_i}$ be the optimal number of class-$i$ customers to be admitted from an arriving batch. Then, we can write inequality (1) for the operators as follows:

$$\kappa^i_{xB_i} R_i + f(x + \kappa^i_{xB_i}) \geq \kappa^i_{x+1B_i} R_i + f(x + 1 + \kappa^i_{x+1B_i}). \quad (5)$$

Since $\kappa^i_{xB_i}$ is the optimal action for the state $x$ and $f(x)$ is non-increasing in $x$, we have:

$$\kappa^i_{xB_i} R_i + f(x + \kappa^i_{xB_i}) \geq \kappa^i_{xB_i} R_i + f(x + \kappa^i_{x+1B_i}), \text{ and}$$

$$\kappa^i_{x+1B_i} R_i + f(x + \kappa^i_{x+1B_i}) \geq \kappa^i_{x+1B_i} R_i + f(x + 1 + \kappa^i_{x+1B_i}).$$

As in the previous proofs, combining these inequalities completes the proof. Therefore, $T_{B_{ADM}} f(x)$ is non-increasing in $x$ if $f(x)$ is a non-increasing function of $x$. As in the pricing operator, the proof for the monotonicity in systems with finite capacity does not change due to the optimality of $\kappa^i_{xB_i}$. 

2
1.2 Upper-Bounded Difference, UBD

In this proof, we show that if \( f(x) \) is an UBD function then \( Tf(x) \) will also be an UBD function for a certain event operator, \( T \). In other words, we prove the following inequality for operator \( T \):

\[
Tf(x) - Tf(x + 1) \leq U,
\]

(6)

whenever \( f(x) - f(x + 1) \leq U \) for some \( U > 0 \).

Since the UBD property of \( f(x) \) is preserved only by the queueing related operators, we only work with the operators \( T_{ARR}, T_{DEP}, T_{CD}, T_{QPRC} \) and \( T_{BADM} \), where the first three operators are not involved with any rewards so that they preserve UBD property for any \( U > 0 \). On the other hand, the operators \( T_{QPRC} \) and \( T_{BADM} \), generate revenue directly, so for these operators we will specify a positive value for \( U \). For this purpose, we will specify a maximum price for \( T_{QPRC} \) and a maximum revenue for \( T_{BADM} \) in their corresponding subsections.

We will prove inequality (6) mainly for systems with infinite capacity. For systems with finite capacity, we focus only on the arrival related operators, since the other operators are not affected by \( K \) (so all the proofs are valid for these operators in systems with finite capacity).

1.2.1 UBD property preserved by \( T_{ARR} \)

We can write inequality (6) for the arrival operator, \( T_{ARR} \), as follows:

\[
a(x)f(x + 1) + [1 - a(x)]f(x) - a(x + 1)f(x + 2) - [1 - a(x + 1)]f(x + 1) \leq U.\]

(7)

Since \( a(x+1) \leq a(x) \) by the monotonicity of the function \( a \), and \( f(x) - f(x+1) \leq U \) by assumption, we can manipulate the inequality (7) as follows:

\[
a(x + 1)[f(x + 1) - f(x + 2)] + [1 - a(x)][f(x) - f(x + 1)] \leq [1 - a(x) + a(x + 1)]U \leq U.
\]

Thus, \( T_{ARR} \) preserves UBD property, if \( f(x) - f(x + 1) \leq U \). For the capacitated systems, we need to consider the state \( x = K - 1 \), for which \( a(K - 1) > 0 \) and \( a(K) = 0 \). Therefore, \( [1 - a(x) + a(x + 1)]U \) is still less than or equal to \( U \). Hence, \( T_{ARR} \) preserves UBD property for the capacitated systems as well.

1.2.2 UBD property preserved by \( T_{DEP} \)

Inequality (6) for this operator can be written as follows:

\[
b(x)f(x - 1) + [1 - b(x)]f(x) - b(x + 1)f(x) - [1 - b(x)]f(x + 1) \leq U,
\]

(8)

which can be manipulated to obtain

\[
b(x)[f(x - 1) - f(x)] + [1 - b(x + 1)][f(x) - f(x + 1)] \leq [1 + b(x) - b(x + 1)]R \leq U,
\]

since \( b(x) \leq b(x + 1) \), and \( f(x) - f(x + 1) \leq U \) by assumption. Therefore, inequality (8) is true, and \( T_{DEP}f(x) - T_{DEP}f(x + 1) \leq U \) if \( f(x) - f(x + 1) \leq U \).
1.2.3 UBD property preserved by $T_{CD}$

Let $\pi_x$ be the optimal service rate for the state $x$. Then, we can write inequality (6) for $T_{CD}$ as follows:

$$-c_{\pi_x} + \pi_x f(x-1) + (1 - \pi_x)f(x) + c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1 - \pi_{x+1})f(x+1) \leq U.$$  (9)

As a result of the optimality of $\pi_{x+1}$, we have:

$$-c_{\pi_x} + \pi_x f(x-1) + (1 - \pi_x)f(x) + c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1 - \pi_{x+1})f(x+1) \leq -c_{\pi_x} + \pi_x f(x-1) + (1 - \pi_x)f(x) + c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1 - \pi_{x+1})f(x+1)$$

$$= \pi_x[f(x-1) - f(x)] + (1 - \pi_x)[f(x) - f(x+1)].$$

Since $f(x) - f(x+1) \leq U$ by assumption, inequality (9) is true, and thus $T_{CD}f(x) - T_{CD}f(x+1) \leq U$.

1.2.4 UBD property preserved by $T_{QPRC}$

As mentioned above, we need to specify a value of $U$ for $T_{QPRC}$. For this purpose, we first assume that allowable prices are bounded, an assumption that is true in all real systems. To be more specific, we set $U = p_{max}$, where $p_{max} = \sup\{p_x : x = 0, 1, 2, \cdots\}$ and $p_x$ is an optimal price in state $x$. Note that $p_{max}$ is finite, since we assume that the prices are bounded. $p_{max}$ is, then, the maximal optimal price. Thus, we assume that function $f$ satisfies $f(x) - f(x+1) \leq p_{max}$ for all $x$.

The operator $T_{QPRC}$ preserves the UBD property of function $f$, only if $f$ is a concave function of $x$. Hence, we assume that $f(x) - f(x+1) \leq f(x+1) - f(x+2)$, which ensures the monotonicity of optimal prices, i.e., $p_x \leq p_{x+1}$ for all $x$.

Now, we write inequality (6) for $T_{QPRC}$ is:

$$\tilde{F}(p_x)[f(x+1) + p_x] + F(p_x)f(x) - \tilde{F}(p_{x+1})[f(x+2) + p_{x+1}] - F(p_{x+1})f(x+1) \leq p_{max}.$$  (10)

Since $p_x \leq p_{x+1}$, $\tilde{F}(p_x) \geq \tilde{F}(p_{x+1})$. Then, we can manipulate the LHS of the inequality to have:

$$\tilde{F}(p_{x+1})[f(x+1) + p_x - f(x+2) - p_{x+1}] + [\tilde{F}(p_x) - \tilde{F}(p_{x+1})][f(x+1) + p_x - f(x+1)] \leq p_{max},$$  (11)

which is always true since $f(x) - f(x+1) \leq p_{max}$ and $p_x \leq p_{x+1} \leq p_{max}$ for all $x \geq 0$. Thus $T_{QPRC}f(x) - T_{QPRC}f(x+1) \leq p_{max}$.

For systems with finite capacity, we need to consider state $x = K-1$ to observe the boundary effects. For this state, only the first line in the LHS of inequality (11) changes, which is still true since $p_x \leq p_{x+1}$. In fact, in finite systems, we can specify $p_{max}$ as $p_{K-1}$ due to the monotonicity of optimal prices.
where \( f \) is a concave function of \( x \) for the operators \( T \). In this proof, we show that if \( f \) is a concave function of \( x \), then the following inequality holds for all possible \((\kappa_x^{iB_i}, \kappa_x^{iB_i+1})\) and for all \( i \):

\[
\kappa_x^{iB_i} R_i + f(x + \kappa_x^{iB_i}) - \kappa_x^{iB_i} R_i - f(x + 1 + \kappa_x^{iB_i}) \leq R_1. 
\]

(12)

It can easily be shown that concavity of \( f \) implies that \( \kappa_x^{iB_i} \) and \( \kappa_x^{iB_i+1} \) optimal decisions in state \( x \) and \( x+1 \), respectively, satisfy either \( \kappa_x^{iB_i} = \kappa_x^{iB_i+1} \) or \( \kappa_x^{iB_i} = \kappa_x^{iB_i+1} + 1 \). Then, it is enough to consider the two cases: \((a,a)\) with \( 0 \leq a \leq B_i \) and \((a,a+1)\) with \( 0 \leq a < B_i \). We rewrite inequality (12) for each case in Table 1. Case II is true since \( R_1 \) is the highest reward offered by the customers, whereas Case I is true by the assumption \( f(x) - f(x+1) \leq f(x+1) - f(x+2) \).

In systems with finite capacity, we need to observe the states \( x \geq K - B_i \) in order to see the boundary effects. However, optimal decisions in these states also will satisfy one of the cases given in Table 1. Hence, \( T_{B,ADM_i} \) preserves UBD property in these systems as well.

### 1.3 Lower-Bounded Difference, LBD

In this proof, we show that if \( f(x) \) is an LBD function then the following inequality holds for the operators: \( T_{C,PRD}, T_{I,PRC} \) and \( T_{B,RT_i} \):

\[
T f(x) - T f(x + 1) \geq L,
\]

(13)

where \( L \) is generally negative. \( T_{C,PRD} \) will preserve the LBD property for any value of \( L \), whereas for the operators \( T_{I,PRC} \) and \( T_{B,RT_i} \), we will give specific values for \( L \). In fact, the proof of LBD of \( T_{C,PRD} \) is straightforward, and so it is omitted.

#### 1.3.1 LBD property preserved by \( T_{I,PRC} \)

The operator \( T_{I,PRC} \) preserves the LBD property of function \( f \), if \( f \) is a concave function of \( x \). Hence, we assume that \( f(x) - f(x + 1) \leq f(x + 1) - f(x + 2) \), which ensure the monotonicity of
optimal prices, so that if \( p_x \) denotes an optimal price in state \( x \), then \( p_{x+1} \leq p_x \). Then, we let \( L = -p_1 \), since \( p_1 \) is the maximal optimal price to be offered by the monotonicity of the prices, i.e., \( p_1 = \max \{ p_x : x \geq 1 \} \). Now, we write inequality (13) for \( T_{LJRC} \) as follows:

\[
\tilde{F}(p_x)[f(x - 1) + p_x] + F(p_x)f(x) - \tilde{F}(p_{x+1})[f(x) + p_{x+1}] - F(p_{x+1})f(x + 1) \geq -p_1.
\]  

(14)

Since \( p_x \geq p_{x+1} \), \( \tilde{F}(p_x) \leq \tilde{F}(p_{x+1}) \). Then, we can manipulate the LHS of the inequality to have:

\[
\tilde{F}(p_x)[f(x - 1) + p_x - f(x) - p_{x+1}] + \tilde{F}(p_{x+1})[f(x) - p_{x+1} - f(x)] + F(p_{x+1})[f(x + 1) - f(x)] \geq -p_1,
\]

which is true since \( f(x) - f(x + 1) \geq -p_1 \) and \( p_{x+1} \leq p_x \leq p_1 \) for all \( x \). Thus, we have \( T_{LJRC}f(x) - T_{LJRC}f(x + 1) \geq -p_1 \).

### 1.3.2 LBD property preserved by \( T_{B_{RT_i}} \)

This proof is very similar to the proof of UBD property preserved by \( T_{B_{ADM_i}} \), so the operator \( T_{B_{RT_i}} \) preserves the LBD property of function \( f \), only if \( f \) is a concave function of \( x \). Hence, we assume that \( f(x) - f(x + 1) \leq f(x + 1) - f(x + 2) \). Moreover, LBD property relates to optimal decisions whenever \( L \) is specified as the reward brought by a class-\( i \) customer. Therefore, we assume that there are \( N \) classes of customers, where class-\( i \) customers are willing to pay \( R_{i} \) per item with \( R_1 \geq R_2 \geq \cdots \geq R_N \), without loss of generality. Then, LBD property is valid for \( L = -R_1 \). Hence, we assume \( f(x) - f(x + 1) \geq -R_1 \), where \( R_1 \) is the maximal reward which can be obtained through class \( 1 \).

Let \( \kappa_{x}^{B_{i}} \) be optimal number of customers to be satisfied from an arriving batch of size \( B_{i} \) in state \( x \). Now, we need to prove the following inequality for all possible \((\kappa_{x}^{B_{i}}, \kappa_{x+1}^{B_{i}})\) and for all \( i \):

\[
\kappa_{x}^{iB_{i}}R_{i} + f(x - \kappa_{x}^{B_{i}}) - \kappa_{x+1}^{iB_{i}}R_{i} - f(x + 1 - \kappa_{x+1}^{B_{i}}) \geq -R_1.
\]  

(15)

It can easily be shown that concavity of \( f \) implies that \((\kappa_{x}^{iB_{i}}, \kappa_{x+1}^{iB_{i}})\) satisfies either \( \kappa_{x}^{B_{i}} = \kappa_{x+1}^{B_{i}} \), or \( \kappa_{x}^{B_{i}} = \kappa_{x+1}^{B_{i}} - 1 \), so it is enough to consider the two cases: \((a, a)\) with \( 0 \leq a \leq B_{i} \) and \((a, a + 1)\) with \( 0 \leq a < B_{i} \). We rewrite inequality (15) for each case in Table 2. Case II is true since \( R_1 \) is the maximal reward, whereas Case I is also true by the assumption \( f(x) - f(x + 1) \geq -R_1 \). Thus, inequality (15) is true for all cases, so that \( T_{B_{RT_i}}f(x) - T_{B_{RT_i}}f(x + 1) \geq -R_1 \).

<table>
<thead>
<tr>
<th>Cases</th>
<th>((\kappa_{x}^{iB_{i}}, \kappa_{x+1}^{iB_{i}}))</th>
<th>Rewritten form of inequality (15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>((a, a))</td>
<td>(f(x - a) - f(x - a + 1) \geq -R_1)</td>
</tr>
<tr>
<td>Case II</td>
<td>((a, a + 1))</td>
<td>(-R_i \geq -R_1)</td>
</tr>
</tbody>
</table>

Table 2: Possible optimal actions in states \( x, x + 1 \) with operator \( T_{B_{RT_i}} \).
1.4 Concavity

In this proof, we show that all operators defined in Section 3 preserves concavity of function $f$. More explicitly, we will show the inequality:

$$\Delta Tf(x) \leq \Delta Tf(x + 1),$$  \hspace{1cm} (16)

for any concave function $f$ for all operators $T$. As in the previous proofs, we show the proofs for systems with infinite capacity, and then consider those with finite capacity when the operators are affected directly. However, here we note that $T_{ARR}$ preserves concavity only when $a(x)$ is non-increasing and convex in $x$.

1.4.1 Concavity preserved by $T_{ARR}$

In this proof, we first note that $a(x)$ is non-increasing and convex in $x$. We can write inequality (16) for $T_{ARR}$ as follows:

$$a(x)f(x + 1) + [1 - a(x)]f(x) - a(x + 1)f(x + 2) - [1 - a(x + 1)]f(x + 1) \leq a(x + 1)f(x + 2) + [1 - a(x + 1)]f(x + 1) - a(x + 2)f(x + 3) - [1 - a(x + 2)]f(x + 3),$$

which can be arranged, by using $a(x + 1) \leq a(x)$, to obtain:

$$[1 - a(x)][f(x) - f(x + 1)] + a(x + 1)[f(x + 1) - f(x + 2)] \leq [1 - a(x + 1)][f(x + 1) - f(x + 2)] + a(x + 2)[f(x + 2) - f(x + 3)].$$

Finally, by adding and subtracting the term $[1 - a(x)][f(x + 1) - f(x + 2)]$ to this inequality, we rearrange the terms by some algebra to obtain:

$$[1 - a(x)][f(x) - f(x + 1)] + [2a(x + 1) - a(x)] [f(x + 1) - f(x + 2)] \leq [1 - a(x)][f(x + 1) - f(x + 2)] + a(x + 2)[f(x + 2) - f(x + 3)].$$  \hspace{1cm} (17)

The first line is true due to the concavity of $f(x)$. Therefore, we focus on the second line. We have the following inequalities as a result of the convexity of the function $a(x)$ and the concavity of $f(x)$:

$$2a(x + 1) - a(x) \leq a(x + 2) \hspace{1cm} (18)$$

$$\Delta f(x) = f(x) - f(x + 1) \leq \Delta f(x + 1) = f(x + 1) - f(x + 2),$$

which imply that inequality (17) is true. Hence, $T_{ARR}$ preserves concavity of a function $f$ in systems with infinite capacity.

For systems with a finite capacity $K$, we need to consider state $K - 2$ to observe the boundary effects. Since the function $a(x)$ is convex for all $x$ with $0 \leq x \leq K$, i.e., even with $a(K) = 0$, inequality (18) will still hold. Notice that this implies $2a(K - 1) \leq a(K - 2)$. Hence, the operator $T_{ARR}$ preserves concavity of a function $f$ in all systems whenever $a(x)$ is convex in $x$ in the whole state space. However, here we note that the regular arrivals which join the system with a fixed probability $a > 0$ cannot satisfy inequality (18) for $x = K - 2$ since $a(x) = a$ for all $x$ with $0 \leq x \leq K - 1$ and $a(K) = 0$. Hence, for regular arrivals, we cannot guarantee that $T_{ARR}$ preserves concavity of a function $f$ in systems with finite capacity.
1.4.2 Concavity preserved by $T_{DEP}$

Similar to the concavity of $T_{ARR}$, we can write and rearrange inequality (16) for $T_{DEP}$ as follows:

\[
\begin{align*}
& b(x)[f(x - 1) - f(x)] \\
& + [1 - 2b(x + 1) + b(x)][f(x) - f(x + 1)] \\
& \leq b(x)[f(x) - f(x + 1)] \\
& + [1 - b(x + 2)][f(x + 1) - f(x + 2)],
\end{align*}
\]

which is true by the concavity of $b(x)$ and $f(x)$ in $x$. Thus, $T_{DEP}f(x)$ preserves concavity of $f(x)$. Here, note that $b(0) = 0$ since there cannot be a departure if there is no customer in the system. In general, the boundary effects due to $x = 0$ should be investigated. However, when $b(x)$ is concave in $x$ for all $x \geq 0$, there is no need for this further investigation. In particular, when $b(x)$ models $m$ parallel servers, it is still concave in $x$, hence the concavity holds for this system as well.

1.4.3 Concavity preserved by $T_{CD}$

Let $\pi_x$ be optimal service rate in state $x$. Then, the concavity inequality of the operator is:

\[
\begin{align*}
& -c_{\pi_x} + \pi_x f(x - 1) + (1 - \pi_x) f(x) \\
& + c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1 - \pi_{x+1}) f(x + 1) \\
& \leq -c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1 - \pi_{x+1}) f(x + 1) \\
& + c_{\pi_{x+2}} - \pi_{x+2} f(x + 1) - (1 - \pi_{x+2}) f(x + 2).
\end{align*}
\]

(19)

We first consider the LHS of inequality (19). Since $\pi_{x+1}$ is the optimal service rate in state $x + 1$, we have:

\[
\begin{align*}
& -c_{\pi_x} + \pi_x f(x - 1) + (1 - \pi_x) f(x) \\
& + c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1 - \pi_{x+1}) f(x + 1) \\
& \leq f(x) - f(x + 1) \\
& + \pi_x ([f(x - 1) - f(x)] - [f(x) - f(x + 1)]) \\
& \leq f(x) - f(x + 1),
\end{align*}
\]

(20)

where the second inequality is true due to some algebra and the third by the concavity of $f(x)$.

Similarly, the RHS of inequality (19) can be manipulated to have:

\[
\begin{align*}
& -c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1 - \pi_{x+1}) f(x + 1) \\
& + c_{\pi_{x+2}} - \pi_{x+2} f(x + 1) - (1 - \pi_{x+2}) f(x + 2) \\
& \geq f(x) - f(x + 1).
\end{align*}
\]

(21)

When inequalities (20) and (21) are combined, it is obvious that inequality (19) is true. Hence, $T_{CD}$ preserves concavity of $f(x)$ in $x$.

Proofs for $T_{CPRD}$, $T_{QPRC}$ and $T_{I_PRC}$ to preserve concavity are similar to this proof. However, while considering the capacitated queues, we need to observe the concavity of $T_{QPRC}$ for the state $x = K - 2$. Since we use the optimal action for the state $x + 1$, i.e., $K - 1$, and it is not affected by the waiting room capacity, so that the foregoing proof is still valid for systems with finite capacity.
\[ \kappa^{iB_i} = (\kappa^{iB_i}_x, \kappa^{iB_i}_{x+1}, \kappa^{iB_i}_{x+2}) \]

<table>
<thead>
<tr>
<th>Cases</th>
<th>Rewritten form of inequality (22)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>((a, a, a))</td>
</tr>
<tr>
<td>Case II</td>
<td>((a+1, a, a))</td>
</tr>
<tr>
<td>Case III</td>
<td>((a+2, a+1, a))</td>
</tr>
<tr>
<td>Case IV</td>
<td>((a+1, a+1, a))</td>
</tr>
</tbody>
</table>

Table 3: Possible optimal actions in states \(x, x+1\) and \(x+2\) with operator \(T_{B,ADM_i}\)

1.4.4 Concavity preserved by \(T_{B,ADM_i}\)

Let \(\bar{\kappa}^{iB_i} = (\kappa^{iB_i}_x, \kappa^{iB_i}_{x+1}, \kappa^{iB_i}_{x+2})\) be an optimal action vector and \(\kappa^{iB_i}_x\) be optimal number of class-\(i\) customers admitted from an arriving batch \(B_i\) in state \(x\). Then, we prove that the batch admission operator will be concave in \(x\) if \(f(x)\) is concave in \(x\). In other words, we show that the following inequality is true for all possible \(\bar{\kappa}^{iB_i}\):

\[
\begin{align*}
    \kappa^{iB_i}_x R_i + f(x + \kappa^{iB_i}_x) & \leq \kappa^{iB_i}_{x+1} R_i + f(x + 1 + \kappa^{iB_i}_{x+1}) \\
    -\kappa^{iB_i}_{x+1} R_i - f(x + 1 + \kappa^{iB_i}_{x+1}) & \leq -\kappa^{iB_i}_{x+2} R_i - f(x + 2 + \kappa^{iB_i}_{x+2}).
\end{align*}
\]

It is enough to consider four different cases for \(\bar{\kappa}^{iB_i}\) as shown in Table 3, because, as mentioned previously, the concavity of \(f(x)\) implies that the optimal number of customers to be admitted in states \(x\) and \(x+1\) can differ at most by 1. We rewrite inequality (22) for each case in Table 3. Case III is obviously true and case I is true due to the concavity of \(f(x)\). In case II, the optimal action in state \(x + 1\) is admitting \(a\) customers so rejecting \((a+1)\)st customer of the arriving batch, which implies that \(f(x + a + 1) \geq R_i + f(x + a + 2)\). Thus, inequality (22) is true in case II. In a similar manner, inequality (22) is also true in case IV by the optimal action in the state \(x + 1\). Therefore, we show that \(T_{B,ADM_i}, f(x)\) is concave in \(x\) when \(f(x)\) is concave in \(x\).

For the capacitated queues, we need to focus on states \(x \geq K - B_i - 1\) to investigate the boundary effect. However, the optimal actions in these states will also fall in one of the categories given in Table 3. Therefore, \(T_{B,ADM_i}, f(x)\) is concave in \(x\) for systems with finite capacity.

The proof of the concavity of \(T_{B,RT}, f(x)\) is similar to this proof.

2 Proof of Lemma 2

In this section, we prove that a certain operator, \(T\), is supermodular with respect to the parameter \(\alpha\) whose effects we would like to observe and \(x\) under certain assumptions, as stated in Lemma 2.

As mentioned in the paper, proving supermodularity requires to consider two systems, one with parameter \(\alpha\) and the other with \(\alpha + \varepsilon\). Here, we simplify the notation introduced in the paper, by denoting functions \(f, a,\) and \(b\) by \(f_{\varepsilon}, a_{\varepsilon},\) and \(b_{\varepsilon}\) after the parameter \(\alpha\) increases by \(\varepsilon\), respectively. The parameter \(\alpha\) will always be clear from the context, so we do not need to indicate it in the functions. Then the inequality for supermodularity of a certain operator, \(T\), is as follows:

\[
\Delta T f(x) \geq \Delta T f_{\varepsilon}(x). \tag{23}
\]
Here, we also need to clarify the definition of the supermodularity of $f$ in $\alpha$ and $x$ when the state space of $x$ is different for systems with parameter $\alpha$ and $\alpha + \varepsilon$. When $\alpha \in \{\lambda, \mu, m\}$ with $m < K$, the two systems have the same state space, whereas increasing $\alpha = K$ by 1 alters the state space of $x$. In the latter case, we define the supermodularity of $f$ in $K$ and $x$ only for $x \leq K$.

### 2.1 Supermodularity preserved by $T_{ARR}$

Let $\alpha$ be in $\{\lambda, \mu, m\}$, and the system have infinite capacity. We first note that $a(x)$ is assumed to be convex in $x$, non-increasing in $\alpha$, and submodular with respect to $\alpha$ and $x$.

We can write inequality (23) for this operator as follows:

$$
\begin{align*}
  a(x)f(x+1) + [1-a(x)]f(x) - a(x+1)f(x+2) - [1-a(x+1)]f(x+1) &\geq
  a_\varepsilon(x)f_\varepsilon(x+1) + [1-a_\varepsilon(x)]f_\varepsilon(x) - a_\varepsilon(x+1)f_\varepsilon(x+2) - [1-a_\varepsilon(x+1)]f_\varepsilon(x+1).
\end{align*}
$$

Now we add and subtract the terms, $a(x)[f_\varepsilon(x) - f_\varepsilon(x+1)]$ and $a(x+1)[f_\varepsilon(x+1) - f_\varepsilon(x+2)]$, to this inequality, and then rearrange it to obtain:

$$
\begin{align*}
  a(x+1)[f(x+1) - f(x+2)] + [1-a(x)][f(x) - f(x+1)] &\geq
  a(x+1)[f_\varepsilon(x+1) - f_\varepsilon(x+2)] + [1-a(x)][f_\varepsilon(x) - f_\varepsilon(x+1)] \quad (24)
\end{align*}
$$

The inequality in the first two lines are true by the supermodularity of $f(x)$ with respect to $\alpha$ and $x$. Now we need to show that the third line also satisfies the inequality. We have the following relations due to the concavity of $f$ and the submodularity of $a$:

$$
\begin{align*}
  0 &\leq f_\varepsilon(x) - f_\varepsilon(x+1) \leq f_\varepsilon(x+1) - f_\varepsilon(x+2) \\
  0 &\leq a(x) - a_\varepsilon(x) \leq a(x+1) - a_\varepsilon(x+1).
\end{align*}
$$

When we combine these inequalities, we have:

$$
0 \leq [a(x) - a_\varepsilon(x)][f_\varepsilon(x) - f_\varepsilon(x+1)] \leq [a(x+1) - a_\varepsilon(x+1)][f_\varepsilon(x+1) - f_\varepsilon(x+2)].
$$

Thus, the third line in inequality (24) is also true. Hence, we complete the proof of the supermodularity of $T_{ARR}f(x)$ with respect to $\alpha$ and $x$ in systems with infinite capacity.

Now we consider systems with a finite capacity $K$, and let $\alpha \in \{\lambda, \mu, m\}$ with $m < K$. We also assume that $a(x)$ is convex in $x$, and constant with respect to $\alpha$. In order to observe the boundary effects, let $x = K - 1$. Then the first line in inequality (24) will be 0, and the second line still satisfies the inequality as before. Finally, the third line is also 0, since $a(x)$ is constant with respect to $\alpha$. Therefore, $T_{ARR}$ preserves supermodularity with respect to $x$ and $\alpha$ whenever $a(x)$ does not depend on $\alpha$. Here, we remind that in finite systems whenever $a(x) = a$ for all $x$, $T_{ARR}$ does not preserve supermodularity, since it cannot preserve concavity (see section 1.4.1).

Finally, we note that supermodularity of a function $f$ with respect to $x$ and $K$ cannot be preserved due to the boundary effects, since the function $a(x)$ inevitably depends on $K$. 

10
2.2 Supermodularity preserved by $T_{DEP}$

The proof for this operator is valid for all $\alpha \in \{\lambda, \mu, m, K\}$ and for both finite and infinite systems. As in the previous proof, we can write and rearrange the supermodularity inequality for this operator as follows:

\[
\begin{align*}
    b(x)[f(x - 1) - f(x)] + [1 - b(x + 1)][f(x) - f(x + 1)] + [b_\varepsilon(x + 1) - b(x + 1)][f_\varepsilon(x) - f_\varepsilon(x + 1)] & \geq b(x)[f_\varepsilon(x - 1) - f_\varepsilon(x)] + [1 - b(x + 1)][f_\varepsilon(x) - f_\varepsilon(x + 1)] + [b_\varepsilon(x) - b(x)][f_\varepsilon(x - 1) - f_\varepsilon(x)].
\end{align*}
\]  

(25)

The first two lines are true due to the supermodularity of $f(x)$. Therefore, if we show that the third line satisfies the inequality, we will complete the proof. Since we assume that $b(x)$ is non-decreasing in $\alpha$, and supermodular with respect to $\alpha$ and $x$, while $f(x)$ is non-increasing and concave in $x$ and supermodular in $\alpha$ and $x$, we have the following inequalities:

\[
\begin{align*}
    0 \leq b_\varepsilon(x) - b(x) & \leq b_\varepsilon(x + 1) - b(x + 1), \\
    0 \leq f_\varepsilon(x) - f_\varepsilon(x + 1) & \leq f_\varepsilon(x + 1) - f_\varepsilon(x + 2).
\end{align*}
\]

When we combine these inequalities, we can obtain:

\[
[b_\varepsilon(x) - b(x)][f_\varepsilon(x - 1) - f_\varepsilon(x)] \leq [b_\varepsilon(x + 1) - b(x + 1)][f_\varepsilon(x) - f_\varepsilon(x + 1)],
\]

which proves that the third line in inequality (25) also satisfies the inequality, so $T_{DEP} f(x)$ is supermodular with respect to $\alpha$ and $x$ under the assumptions mentioned in Lemma 2. In particular, for $m$ identical parallel servers $b(x)$ is always non-decreasing in $m$ and supermodular with respect to $m$ and $x$, so that $T_{DEP} f(x)$ is supermodular with respect to $m$ and $x$.

2.3 Supermodularity preserved by $T_{CD}$

The proof for this operator is valid for all $\alpha \in \{\lambda, \mu, K\}$ and for both finite and infinite systems. Let $\pi_x$ and $\tilde{\pi}_x$ be the optimal service rates for the state $x$ before and after the parameter $\alpha$ increases, respectively. Then, we show that the following supermodularity inequality is true for $T_{CD}$.

\[
\begin{align*}
    c_{\pi_x} + \pi_x f(x - 1) + (1 - \pi_x) f(x) - c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1 - \pi_{x+1}) f(x + 1) & \geq c_{\tilde{\pi}_x} + \tilde{\pi}_x f_\varepsilon(x - 1) + (1 - \tilde{\pi}_x) f_\varepsilon(x) - c_{\tilde{\pi}_{x+1}} - \tilde{\pi}_{x+1} f_\varepsilon(x) - (1 - \tilde{\pi}_{x+1}) f_\varepsilon(x + 1).
\end{align*}
\]  

(26)

As a result of the optimality of $\pi_x$ for the state $x$, the LHS of the inequality can be bounded from below as follows:

\[
\begin{align*}
    c_{\pi_x} + \pi_x f(x - 1) + (1 - \pi_x) f(x) - c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1 - \pi_{x+1}) f(x + 1) & \geq c_{\tilde{\pi}_x} + \tilde{\pi}_x f(x - 1) + (1 - \tilde{\pi}_x) f(x) - c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1 - \pi_{x+1}) f(x + 1) \\
    & = c_{\tilde{\pi}_x} - c_{\pi_{x+1}} + \tilde{\pi}_x [f(x - 1) - f(x)] + (1 - \pi_{x+1}) [f(x) - f(x + 1)],
\end{align*}
\]  

(27)
where the equality follows by some algebra. Similarly, we can obtain the following for the RHS of inequality (26) due to the optimality of \( \tilde{\pi}_{x+1} \) and by some algebra:

\[
\begin{align*}
&c_{\tilde{\pi}_{x}} + \tilde{\pi}_{x} f_{c}(x-1) + (1 - \tilde{\pi}_{x}) f_{c}(x) \\
&- c_{\tilde{\pi}_{x+1}} - \tilde{\pi}_{x+1} f_{c}(x) - (1 - \tilde{\pi}_{x+1}) f_{c}(x+1) \leq c_{\tilde{\pi}_{x}} - c_{\tilde{\pi}_{x+1}} + \tilde{\pi}_{x} [f_{c}(x-1) - f_{c}(x)] \\
&\quad + (1 - \tilde{\pi}_{x+1}) [f_{c}(x) - f_{c}(x+1)].
\end{align*}
\] (28)

Inequalities (27) and (28) together with the supermodularity of the function \( f \) imply that inequality (26) is true. Thus, we complete the proof of the supermodularity of \( T_{C,D} f(x) \) with respect to \( \alpha \) and \( x \).

Proofs of the supermodularity of \( T_{C,PRD}, T_{Q,PRC} \) and \( T_{L,FRC} \) are similar to this proof.

### 2.4 Supermodularity preserved by \( T_{B,ADM} \)

We first consider systems with infinite capacity, and let \( \alpha \in \{\lambda, \mu, m\} \). We denote by \( \kappa^{IB_i} = (\kappa^{IB_i}_x, \kappa^{IB_i}_{x+1}, a^{IB_i}_x, i^{IB_i}_x x + 1) \) the optimal action vector, where \( \kappa^{IB_i}_x \) and \( \tilde{\kappa}^{IB_i}_x \) are the optimal number of customers to be admitted from an arriving batch in state \( x \) before and after the parameter \( \alpha \) increases, respectively. Then, we show that the following supermodularity inequality is true for the batch admission operator:

\[
\begin{align*}
\kappa^{IB_i}_x R_i + f(x+\kappa^{IB_i}_x) - \kappa^{IB_i}_{x+1} R_i - f(x+1+\kappa^{IB_i}_{x+1}) &\geq \tilde{\kappa}^{IB_i}_x R_i + f(x+\tilde{\kappa}^{IB_i}_x) - \tilde{\kappa}^{IB_i}_{x+1} R_i - f(x+1+\tilde{\kappa}^{IB_i}_{x+1}).
\end{align*}
\] (29)

We have to consider all possible optimal action vectors to prove that supermodularity is preserved by \( T_{B,ADM} \). We know that \( \kappa^{IB_i}_x \) and \( \kappa^{IB_i}_{x+1} \) can differ at most by 1 due to concavity of \( f \). Moreover, again by concavity of \( f \), if \( \kappa^{IB_i}_x = \kappa^{IB_i}_{x+1} \), we either have \( \kappa^{IB_i}_x = \kappa^{IB_i}_{x+1} = 0 \) or \( \kappa^{IB_i}_x = \kappa^{IB_i}_{x+1} = B_i \).

The supermodularity of \( f \) with respect to \( \alpha \) and \( x \), on the other hand, implies that \( \kappa_x \leq \tilde{\kappa}^{IB_i}_x \) for all \( x \). Hence, it is enough to consider the following cases: \((0, 0, 0, 0), (0, 0, a + 1, a), (0, 0, B_i, B_i), (a + 1, a, d + 1, d), (a + 1, a, B_i, B_i) \) and \((B_i, B_i, B_i, B_i)\), where \( 0 \leq a \leq B_i - 1 \) and \( a \leq d \leq B_i - 1 \).

Table 4 presents inequality (29) for all these six cases. Case IV is obviously true, while Cases I and VI are true by the supermodularity of \( f(x) \). In Case II, it is optimal to reject the entire batch in state \( x \) of the system with parameter \( \alpha \), so that \( f(x) - f(x+1) \geq R_i \), which shows that the inequality holds in Case II. Similarly, the optimal action of Case V is to admit the whole batch in state \( x + 1 \) for the system with parameter \( \alpha + \varepsilon \), which implies that \( B_i R_i + f_{c}(x+B_i+1) \geq (B_i-1) R_i + f_{c}(x+B_i) \).

Since this inequality coincides with the expression in Table 4, inequality (29) is true in Case V as well. Finally, in Case III optimal actions in state \( x \) of the system with parameter \( \alpha \) and in \( x + 1 \) of the system with parameter \( \alpha + \varepsilon \) ensure that inequality (29) holds. Thus, we show that inequality (29) is true for all of the six cases and \( T_{B,ADM} f(x) \) is supermodular with respect to \( \alpha \) and \( x \) if \( f(x) \) is concave in \( x \) and supermodular with respect to \( \alpha \) and \( x \).

In systems with finite capacity, the decisions in \( x \) and \( x + 1 \) of systems with parameter \( \alpha \) and \( \alpha + \varepsilon \) will also satisfy one of the six cases we consider for \( \alpha \in \{\lambda, \mu, m, K\} \). Hence, \( T_{B,ADM} \) preserves supermodularity in finite systems as well.

The proof of \( T_{B,RT_i} \) is similar to this proof.
Rewritten form of inequality (29)

Table 4: Possible optimal actions in states $x$ and $x + 1$ in systems with parameters $\alpha$ and $\alpha + \varepsilon$

### 3 Counterexamples for Supermodularity and Submodularity

In the paper, we commented on how an increase in $K$ affects the controlling operators, i.e., $T_{Q_{PRC}}, T_{B_{\text{ADM}}}$, and $T_{C_{PRD}}$, and the un-controlling operator, i.e., $T_{ARR}$. The controlling operators cannot preserve submodularity, whereas un-controlling one cannot preserve supermodularity. Here, we present two simple counter examples for operators $T_{B_{\text{ADM}}}$ and $T_{ARR}$. Similar examples can be produced for all the above operators.

We first consider $T_{B_{\text{ADM}}}$ with $B_i = 1$, so each batch consists of only one customer. We set:

$$f_K(x) = f_{K+1}(x) = 0 \quad \forall x.$$  

Therefore, $f_K(x)$ is both non-increasing and concave in $x$, and submodular in $K$ and $x$. Applying the operator $T_{B_{\text{ADM}}}$ gives:

$$T_{B_{\text{ADM}}} f_K(x) = R_i \quad \forall x = 0, \ldots, K - 1, \quad \& \quad T_{B_{\text{ADM}}} f_K(K) = 0,$$

$$T_{B_{\text{ADM}}} f_{K+1}(x) = R_i \quad \forall x = 0, \ldots, K, \quad \& \quad T_{B_{\text{ADM}}} f_{K+1}(K + 1) = 0.$$  

Now we can check the inequality for $T_{B_{\text{ADM}}}$ to preserve the submodularity for state $x = K - 1$:

$$T_{B_{\text{ADM}}} f_K(K - 1) - T_{B_{\text{ADM}}} f_K(K) = R_i \leq T_{B_{\text{ADM}}} f_{K+1}(K - 1) - T_{B_{\text{ADM}}} f_{K+1}(K) = 0,$$

which does not hold, so that the submodularity is not preserved.

We now consider $T_{ARR}$ with $a(x) = 1$ whenever the finite room has at least one empty space, and we set:

$$f_K(x) = f_{K+1}(x) = K + 1 - x \quad \forall x.$$  

Therefore, $f_K(x)$ is both non-increasing and concave in $x$, and supermodular in $K$ and $x$. Applying the operator $T_{ARR}$ gives:

$$T_{ARR} f_K(x) = K - x \quad \forall x = 0, \ldots, K - 1, \quad \& \quad T_{ARR} f_K(K) = 1,$$

$$T_{ARR} f_{K+1}(x) = K - x \quad \forall x = 0, \ldots, K, \quad \& \quad T_{ARR} f_{K+1}(K + 1) = 0.$$  

Now we can check the inequality for $T_{ARR}$ to preserve the supermodularity for state $x = K - 1$:

$$T_{ARR} f_K(K - 1) - T_{ARR} f_K(K) = 0 \geq T_{ARR} f_{K+1}(K - 1) - T_{ARR} f_{K+1}(K) = 1,$$

which does not hold, so that the supermodularity is not preserved.
4 Proof of Lemma 4

In this proof, we show that if \( f(x) \) has some structural properties then, \( Tf(x) - f(x) \) will be either non-increasing or non-decreasing in \( x \) according to the characteristics of the operator \( T \). As we mentioned in the paper, while considering queueing problems, \( Tf(x) - f(x) \) is non-decreasing in \( x \) for the departure-related operators and non-increasing in \( x \) for the arrival-related operators. On the other hand, \( Tf(x) - f(x) \) is non-increasing in \( x \) for the production operator and non-decreasing in \( x \) for the arrival-related operators.

We use the following inequalities to denote the structure of \( Tf(x) - f(x) \): Inequality \((30)\) represents that \( Tf(x) - f(x) \) is non-decreasing in \( x \), whereas inequality \((31)\) represents that \( Tf(x) - f(x) \) is non-increasing in \( x \).

\[
\begin{align*}
Tf(x) - f(x) &\leq Tf(x + 1) - f(x + 1), \\
Tf(x) - f(x) &\geq Tf(x + 1) - f(x + 1).
\end{align*}
\]

4.1 \( T_{ARR}f(x) - f(x) \)

We need to show that \( T_{ARR}f(x) - f(x) \) is non-increasing in \( x \). However, as we mentioned in the paper, this property holds for the arrival operator when \( a(x) \) is constant and the buffer capacity is infinite. Then, under these assumptions, inequality \((31)\) for \( T_{ARR} \) becomes:

\[ a[f(x + 1) - f(x)] \geq a[f(x + 2) - f(x + 1)], \]

which is true by the concavity of \( f(x) \), so that the proof is completed.

4.2 \( T_{DEP}f(x) - f(x) \)

We will prove \( T_{DEP}f(x) - f(x) \) to be non-decreasing in \( x \). Inequality \((30)\) for the departure operator, \( T_{DEP} \), can be written as:

\[ b(x)[f(x - 1) + (1 - b(x))f(x) - f(x)] \leq b(x + 1)f(x) + (1 - b(x + 1))f(x + 1) - f(x + 1). \]

When we arrange this inequality, we have:

\[ b(x)[f(x - 1) - f(x)] \leq b(x)[f(x) - f(x + 1)] + [b(x + 1) - b(x)][f(x) - f(x + 1)], \]

which is true by the concavity of \( f(x) \), and due to the assumptions \( b(x) \leq b(x + 1) \) and \( f(x) \geq f(x + 1) \). Thus, the proof is complete.

As we mentioned in the previous proofs, the foregoing proof is still valid for the capacitated systems because the departure related operators are not affected by the boundary effects.

4.3 \( T_{CD}f(x) - f(x) \)

Let \( \pi_x \) be the optimal service rate for the state \( x \). As in the departure operator, \( T_{CD}f(x) - f(x) \) is non-decreasing in \( x \) intuitively and we write inequality \((30)\) for this operator as:

\[ c_{\pi_x} + \pi_x f(x - 1) + (1 - \pi_x) f(x) - f(x) \leq c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1 - \pi_{x+1}) f(x + 1) - f(x + 1). \]
Now we consider the RHS of this inequality:

\[
\begin{align*}
    c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1 - \pi_{x+1}) f(x + 1) - f(x + 1) & \geq c_{\pi_{x}} + \pi_{x} f(x) + (1 - \pi_{x}) f(x + 1) - f(x + 1) \\
    & = c_{\pi_{x}} f(x) - f(x + 1) + \pi_{x} f(x) - f(x) \\
    & \geq c_{\pi_{x}} f(x) - f(x),
\end{align*}
\]

where the first inequality follows from the optimality of \(\pi_{x+1}\) and the second is due to the concavity of \(f\), while the equalities follow by some algebra. Hence, inequality (32) is true, and \(T_{CD} f(x) - f(x)\) is non-decreasing in \(x\).

### 4.4 \(T_{PRD} f(x) - f(x)\)

We will prove that \(T_{PRD} f(x) - f(x)\) is non-increasing in \(x\). The proof is similar to the previous one. Let \(\pi_{x}\) be the optimal service rate in state \(x\). Then, inequality (31) becomes:

\[
\begin{align*}
    c_{\pi_{x}} + \pi_{x} f(x) + (1 - \pi_{x}) f(x) - f(x) & \geq c_{\pi_{x+1}} + \pi_{x+1} f(x + 2) + (1 - \pi_{x+1}) f(x + 1) - f(x + 1),
\end{align*}
\]

(33)

Now we consider the LHS:

\[
\begin{align*}
    c_{\pi_{x}} f(x + 1) + (1 - \pi_{x}) f(x) - f(x) & \geq c_{\pi_{x+1}} f(x + 1) + (1 - \pi_{x+1}) f(x) - f(x) \\
    & \geq c_{\pi_{x+1}} + \pi_{x+1} f(x + 2) + (1 - \pi_{x+1}) f(x + 1) - f(x + 1),
\end{align*}
\]

where the first inequality is due to the optimality of \(\pi_{x}\) and the second due to the concavity of \(f\). This proves inequality (33). Thus, \(T_{PRD} f(x) - f(x)\) is non-decreasing in \(x\).

### 4.5 \(T_{PRC} f(x) - f(x)\)

We show that \(T_{PRC} f(x) - f(x)\) is non-increasing in \(x\), so that \(T_{PRC} f(x)\) should satisfy inequality (31). Letting \(p_{x}\) be the optimal price in state \(x\), we have:

\[
\begin{align*}
    \tilde{F}(p_{x})[f(x + 1) + p_{x}] + F(p_{x}) f(x) - f(x) & \geq \tilde{F}(p_{x+1})[f(x + 2) + p_{x+1}] + F(p_{x+1}) f(x + 1) - f(x + 1),
\end{align*}
\]

(34)

We consider the LHS of this inequality:

\[
\begin{align*}
    \tilde{F}(p_{x})[f(x + 1) + p_{x}] + F(p_{x}) f(x) - f(x) & \geq \tilde{F}(p_{x+1})[f(x + 1) + p_{x+1}] + F(p_{x+1}) f(x) - f(x) \\
    & \geq \tilde{F}(p_{x+1})[f(x + 2) + p_{x+1}] + F(p_{x+1}) f(x + 1) - f(x + 1),
\end{align*}
\]

where the first inequality follows by the optimality of \(p_{x}\) and the second by the concavity of \(f\), while the equalities are due to some algebra. Hence, inequality (34) is true, so that \(T_{PRC} f(x) - f(x)\) is non-decreasing in \(x\).

In the capacitated case, we need to consider inequality (34) for state \(x = K - 1\). Then the RHS of (34) becomes 0, so that it is enough to show that the LHS of (34) is non-negative. By simple algebra, the LHS becomes \(\tilde{F}(p_{K-1})[f(K) + p_{K-1} - f(K - 1)]\), which is clearly non-negative due to the optimality of \(p_{K-1}\). Hence, inequality (34) is also true for the capacitated queues.
\[
\begin{array}{|c|c|c|}
\hline
\text{Cases} & \kappa^{iB_i} = (\kappa^{iB_i}_x, \kappa^{iB_i}_{x+1}) & \text{Rewritten form of inequality (36)} \\
\hline
\text{Case I} & (0,0) & 0 \geq 0 \\
\text{Case II} & (a+1,a) & R_i \geq f(x) - f(x+1) \\
\text{Case III} & (B,B) & f(x+B_i) - f(x) \geq f(x+B_i+1) - f(x+1) \\
\hline
\end{array}
\]

Table 5: Possible optimal actions in states \(x\) and \(x+1\)

### 4.6 \(T_{I,P_{RC}}f(x) - f(x)\)

We will show that \(T_{I,P_{RC}}f(x) - f(x)\) is non-decreasing in \(x\). Therefore, we write inequality (30) for this operator as follows:

\[
\bar{F}(p_x)[f(x-1) + p_x] + F(p_x)f(x) - f(x) \leq \bar{F}(p_{x+1})[f(x) + p_{x+1}] + F(p_{x+1})f(x+1) - f(x+1),
\]

where \(p_x\) is the optimal price in state \(x\). As in the proof for the operator \(T_{Q,P_{RC}}f(x)\), we use the optimality of \(p_{x+1}\) and the concavity of \(f\), as well as some algebra to manipulate the RHS of this inequality as follows:

\[
\bar{F}(p_{x+1})[f(x) + p_{x+1}] + F(p_{x+1})f(x+1) - f(x+1) \geq \bar{F}(p_x)p_x + \bar{F}(p_x)[f(x) - f(x+1)] + F(p_x)f(x) - f(x),
\]

which proves inequality (35), so that \(T_{Q,P_{RC}}f(x) - f(x)\) is non-decreasing in \(x\).

### 4.7 \(T_{B,ADM_i}f(x) - f(x)\)

Let \(\kappa^{iB_i}_x\) be the optimal number of customers admitted from an arriving batch in state \(x\). Then, we prove that \(T_{B,ADM_i}f(x) - f(x)\) is non-increasing in \(x\). In other words, we show that the following inequality is true for all possible \((\kappa^{iB_i}_x, \kappa^{iB_i}_{x+1})\):

\[
\kappa^{iB_i}_x R_i + f(x + \kappa^{iB_i}_x) - f(x) \geq \kappa^{iB_i}_{x+1} R_i + f(x+1 + \kappa^{iB_i}_{x+1}) - f(x+1)
\]  

(36)

Due to the concavity of \(f(x)\) and it is enough to consider three cases for optimal actions \((\kappa^{iB_i}_x, \kappa^{iB_i}_{x+1})\): \((0,0), (a+1,a)\) and \((B_i,B_i)\). We rewrite inequality (36) for each case in Table 5. Case I is obviously true, and case III is true due to the concavity of \(f(x)\). In case II, it is optimal to admit \(a+1\) customers from an arriving batch in state \(x\), which implies that \(R_i \geq f(x+a) - f(x+a+1)\). Moreover, by the concavity of \(f(x)\), we have that \(f(x) - f(x+1) \leq f(x+a) - f(x+a+1)\) for all \(a > 0\). When we combine these two inequalities, we obtain that \(R_i \geq f(x) - f(x+1)\) and thus, inequality (36) is true in case II. Therefore, \(T_{B,ADM_i}f(x) - f(x)\) is non-increasing in \(x\) when \(f(x)\) is concave in \(x\).

For the capacitated queues, we need to focus on the states \(x \geq K - B_i\) in order to investigate the boundary effect. For these states, Case III is not possible because admitting \(B_i\) customers in state \(x+1\) is not feasible, which leaves only Cases I and II, whose proofs are the same as above.

The proof for \(T_{B,RT_i}\) is similar to this proof.