Submitted to

Effects of system parameters on the optimal cost and policy in a class of multi-dimensional queueing control problems

Samuel Vercraene
INSA-Lyon, DISP, Villeurbanne F-69621, France, samuel.vercraene@insa-lyon.fr

Jean-Philippe Gayon
G-SCOP, Grenoble-INP, 46 avenue Félix Viallet, 38031 Grenoble Cedex 1, France, jean-philippe.gayon@grenoble-inp.fr

Fikri Karaesmen
Koç University, Sariyer, 34450, Istanbul, Turkey, fkaraesmen@ku.edu.tr

We consider a class of Markov Decision Processes frequently employed to model queueing and inventory control problems. For these problems, we explore how changes in different system input parameters (transition rates, costs, discount rates etc.) affect the optimal cost and the optimal policy when the state space of the problem is multi-dimensional. To address a large class of problems, we introduce two generic dynamic programming operators to model different types of controlled events. For these operators, we derive sufficient conditions to propagate monotonicity and supermodularity properties of the value function. These properties allow to predict how changes in system input parameters affect the optimal cost and policy. Finally, we explore the case when several parameters are changed at the same time.

Key words: Markov decision process, optimal policy, sensitivity analysis, event based dynamic programing.

1. Introduction

Many interesting queueing and inventory control problems can be modeled by continuous-time Markov Decision Processes. A lot of research effort has gone into the investigation of optimal policy structure in such problems. These problems are especially challenging when the state space of the problem is multi-dimensional as in the case of queueing systems with multiple queues or inventory systems with multiple products for example. Even though each new model is different and presents new challenges, a powerful approach called event-based dynamic programming proposed by Koole (1998, 2006) provides a way of establishing results on the optimal policy structure for a certain class of models in a unified manner.

Our main objective in this paper is to further explore optimal policy structure in multi-dimensional queueing and inventory control problems. In particular, we investigate how the optimal cost (or reward) and the optimal policy change when problem input parameters change. The input parameters in question are the transition rates (or probabilities) governing the controlled Markov chain and the financial parameters such as costs, rewards or discount rates. We are seeking answers to questions such as: how does the optimal cost and the optimal policy change when the customer arrival rate or the waiting cost increases in a controlled queueing system? To answer such questions, we construct a systematic approach that first builds on earlier results to infer optimal policy structure and then develop a methodology to explore the effects of varying input parameters.

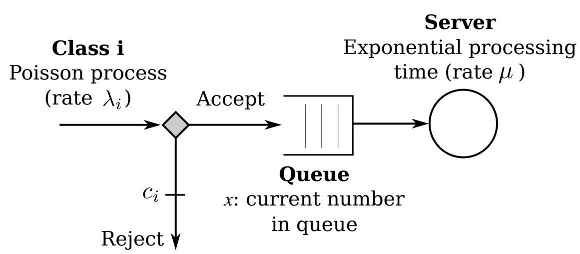
Characterizing and understanding optimal policy structure in queueing and inventory control has received considerable attention. One reason for this emphasis is that understanding the theoretically optimal policy for a simplified model of reality is useful in designing near optimal working policies. This is certainly the case if the optimal policy is not completely defined by a few parameters as in most multi-dimensional problems. In this case, the policy designer would greatly benefit from information about how the working policy parameters should be adjusted and the cost implications of such adjustments when input conditions change. Our methodology addresses this question at a fairly general level.

This paper makes the following contributions. First, we provide a general description of the multi-dimensional problem. Second, we formalize the definition of required structural properties and the condition under which they hold. Finally, we present a systematic approach based on this formalism to express and verify the required conditions.

In what follows, we present two examples that will be used throughout the paper to illustrate our approach and results. The first example explores a well-known admission control problem where the state space is single dimensional. Here, we can contrast the new approach with the existing methodology and demonstrate how some of the earlier results can be complemented. The second example is based on the make-to-stock version of a tandem queueing system. In this case, the state space is multi dimensional and the optimal policy structure is more complicated.

Example 1: Admission control. Consider the following admission control problem with n classes of customers, adapted from Stidham (1985) and illustrated in Figure 1. Customers of class$-i$ arrive according to a Poisson process with rate λ_i and are either accepted or rejected at cost c_i. Once accepted, customers are not differentiated and the service time of the single server is exponentially distributed with rate μ. The state is the number $x \in \mathbb{Z}^+$ of customers in the system. The waiting cost is h per customer per unit of time. The objective is to minimize the expected discounted/average cost over an infinite horizon (with discount rate η). Stidham (1985) proves that the optimal policy is a threshold policy where a class-i customer is accepted if and only if $x < t_i$. In addition $t_i \leq t_j$ if $c_i \leq c_j$. Çil et al. (2009) establish that t_i is increasing in μ and decreasing in λ. We will complete these results by showing that the thresholds are increasing with h and c_i and decreasing with η (see Section 6).

Figure 1 **Admission control model with n classes of customers**



We are also interested by the effect of a parameter change on the optimal cost. For a single class of customers, Figure 2a shows that the optimal cost is increasing and concave in the holding cost h. We also observe that the optimal cost is linear in each interval where the optimal threshold is constant. We will prove these results, among others, in Section 6. Figure 2b provides an example where the optimal cost is decreasing in μ but is neither concave nor convex. However the optimal cost is convex in each interval where the optimal threshold is constant.

Table 1 illustrates what we call compensation between several parameters. We consider three instances where we vary simultaneously the arrival rates of three classes of customers, in such a way that the sum of arrival rates remains equal to 1.8. We observe that the optimal thresholds of Instance 2 are smaller than in Instance 1. However we can not order similarly the thresholds of Instance 1 and Instance 3. In Section 7, we will provide conditions under which we can predict the increase or decrease of the optimal thresholds and costs.

Example 2 : Tandem queue. This second example is to illustrate the effect of changing parameters on the optimal policy in a two-dimensional problem. The make-to-stock tandem queue model of Veatch and Wein (1992, 1994) is illustrated in Figure 3. Servers M_i produces items one by one, with exponentially distributed processing times (rate μ_i). Produced items at server i are held in

Table 1 **Compensation ($\mu = 1$, $h = 1$, $c_1 = 5$, $c_2 = 10$, and $c_3 = 15$)**

Instance	λ_1	λ_2	λ_3	t_1	t_2	t_3	Optimal cost
1	0.6	0.6	0.6	1	2	5	8.98
2	0.1	0.7	1	0	1	4	12.1
3	0.1	1.6	0.1	0	3	7	10.6

buffer B_i. Demand, if not immediately satisfied, is backlogged in buffer B_d. The state of the system is described by (x_1, x_2) with x_1 the number of work-in-process products in B_1 and x_2 the number of serviceable products in B_2 minus the number of backlogged demand in B_d. The system incurs a holding cost h_i per unit of time and unit of product in buffer B_i and a backorder cost b per unit of time and unit of waiting demand. The objective is to minimize the expected discounted/average cost over an infinite horizon (with discount rate η). Veatch and Wein (1992) prove that the optimal policy exists and is a state dependent base stock policy defined by two switching curves: Produce at M_i iff $x_2 < s_i(x_1)$, for $i = 1, 2$.

Figures 4a and 4b show the influence of the demand rate λ and the service rate μ_2 on the optimal switching curves. We observe that λ has a monotonic effect on the switching curves. The switching curve s_1 (resp. s_2) for $\lambda = 1$ is systematically below the one for $\lambda = 1.1$. We will prove in Section 6 that this result holds in general. On the contrary we observe that μ_2 has a non-monotonic effect on switching curve s_1: The curve for $\mu_2 = 1.2$ crosses the curve for $\mu_2 = 2$.

The rest of the paper is organized as follows. Next section reviews the literature and our contributions. Section 3 presents the class of problems and operators under consideration. Section 4 introduces several properties of value functions and state spaces. Section 5 presents our approach and main results to study the effect of changing parameters on the optimal cost and policy. Section 6 applies our results to the admission control and tandem queue problems. Section 7 exhibits compensation phenomenons when several parameters are changed simultaneously.

2. Literature review

Structure of the optimal policy. In a number of queueing control problems, the optimal policy can be described by thresholds, switching curves, or hyperplanes. Several papers develop general approaches for deriving structural properties of the optimal policy (Weber and Stidham 1987, Veatch and Wein 1992, Smith and McCardle 2002, Zhuang and Li 2012). In particular Koole (1998, 2006) presents the so-called event-based dynamic programming framework to study queueing control problems. In this framework, an operator is associated to each type of event (demand arrival, end of service, processor failure, etc) and can be studied individually. To characterize the structure of the optimal policy, some properties of the value function such as monotonicity, convexity/concavity and supermodularity/submodularity is needed. If each individual event operator

Figure 2 **Effect of parameters on the optimal cost**

(a) Concavity and piecewise linearity in h ($n=1$, $\lambda_1=0.6$, $c_1=5$, $\mu=1$, $\eta=0$)

(b) Piecewise convexity in μ ($n=1$, $\lambda_1=0.6$, $c_1=5$, $h=1$, $\eta=0$).

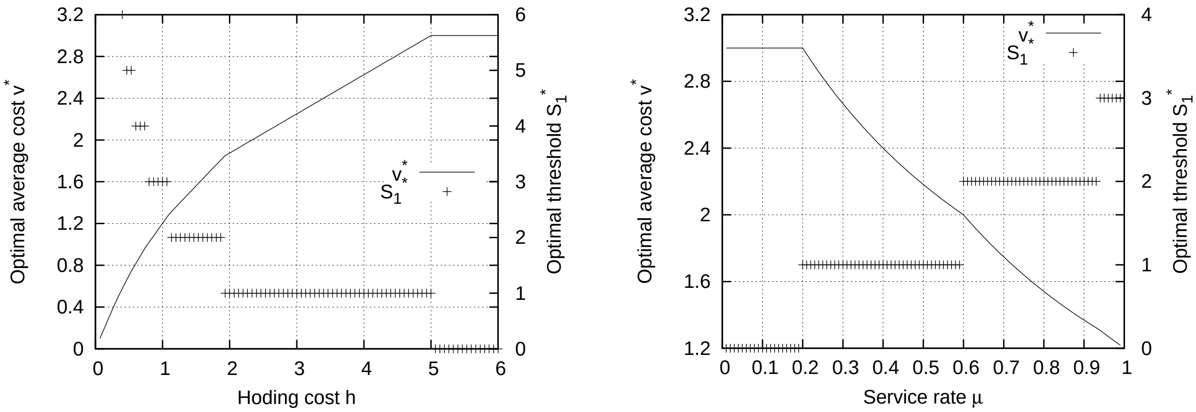


Figure 3 **Tandem make-to-stock queue model**

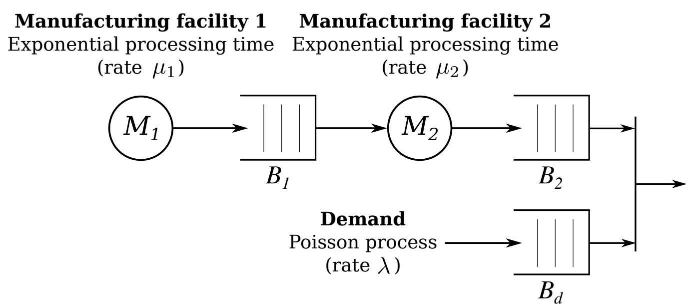
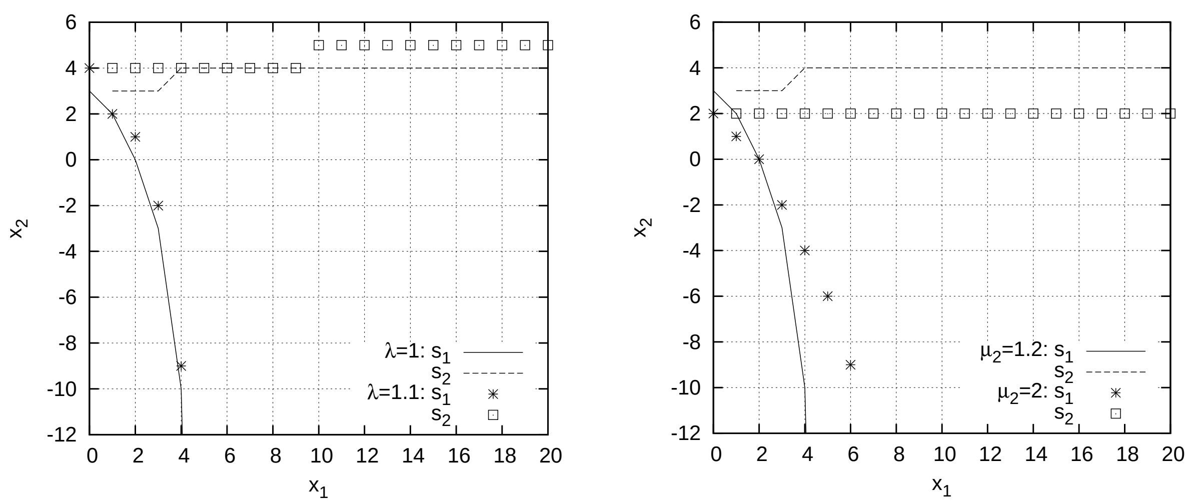


Figure 4 **Effect of parameters on the optimal policy**

(a) Monotonic effect of λ ($\mu_1=2$, $\mu_2=1.2$, $h_1=1$, $h_2=2$, $b=4$, $\eta=0.1$)

(b) Non monotonic effect of μ_2 ($\lambda=1$, $\mu_1=2$, $h_1=1$, $h_2=2$, $b=4$, $\eta=0.1$)



propagates a desired property, then the optimal value function, which is a composition of different individual operators, will also possess this property.

Effect of system parameters on the optimal cost. Many papers investigate numerically the effect of some problem input parameters on the optimal cost and the optimal policy in specific queueing control problems. There is a rich literature on how average performance measures change in terms of the input parameters for uncontrolled queueing systems. However, very few papers investigate this question in the context of controlled queueing systems from a theoretical point of view. Müller (1997) compares the optimal value function of discrete time Markov decision processes that differ only in their transition probabilities. His method requires establishing some complex and restrictive stochastic orders and exploiting some (a priori) known properties of the value function. Koole (2006) employs the event-based dynamic programming framework to address the problem but the relevant monotonicity and convexity results of the optimal value function are obtained for operators with no decision (arrival, departure, etc).

Effect of system parameters on the optimal policy. The implications on the optimal policy of changes in input parameters is less studied. Some papers such as Ku and Jordan (1997), Gans and Savin (2007), Aktaran-Kalaycı and Ayhan (2009) investigate the policy effects of changing input parameters in specific queueing control examples. Zhuang and Li (2012) propose a method based on the general property of multimodularity to establish structural results on the optimal policy for a class of problems. For a specific example, they show that multimodularity also enables obtaining monotonicity results on the parameters of the optimal policy with respect to the input parameters. In contrast, Çil et al. (2009) develop a general approach using the framework of event-based dynamic programming to systematically study the effects of changing input parameters. However, their analysis is mostly restricted to problems where the state space is single dimensional and the optimal policy can be described by thresholds. Their results have been employed in several recent papers for different applications (see e.g. Aydin et al. (2009), Zerhouni et al. (2013), Benjaafar et al. (2010), Satir et al. (2012), Ozkan et al. (2013)).

Contributions. We systematically explore the effects of changing input parameters in a class of queueing control problems. Our main contributions can be summarized as follows. First, we extend the modeling scope in Çil et al. (2009) to problems where the state space is multi dimensional, by introducing two generic operators that include as special cases many commonly used operators from the literature. Second, we study the monotonicity and convexity/concavity of the optimal cost in addition to the monotonicity of the optimal policy with respect to transition rate parameters. Third, we investigate the effect of changing the cost/reward parameters and discount rate. We also exhibit compensation phenomena when several parameters are changed at the same time. Overall, we formalize and generalize the proof methodology of Çil et al. (2009) by using Boolean equations so that it can be applied to a richer class of problems. We believe that our approach opens interesting perspectives on the automation of proofs of structural properties.

3. The operators

Consider a continuous-time MDP with the objective to minimize the expected discounted cost over an infinite horizon with discount rate η. Our results can be easily adapted to maximization problems, finite horizon problems or average cost problems.

The state is an m_1-dimensional vector $\mathbf{s_1} \in \mathcal{S}_1 \in \mathbb{Z}^{m_1}$ where $\mathbb{Z}$ is the set of all integers. We assume that the system parameters (transition rates, costs, discount rate) can be summarized in a m_2-dimensional vector $\mathbf{s_2} \in \mathcal{S}_1 \subset \mathbb{R}^{m_2}$ where $\mathbb{R}$ is the set of real numbers. We can aggregate $\mathbf{s_1}$ and $\mathbf{s_2}$ in an (m_1+m_2)-dimensional vector $\mathbf{x} = (\mathbf{s_1}, \mathbf{s_2}) \in \mathcal{X} = \mathcal{S}_1 \times \mathcal{S}_2$. In the rest of the paper, vector $\mathbf{x}$ will be referred to as the *system state* while $\mathbf{s_1}$ will be simply called the *state*. The set $\mathcal{X}$ will be referred to as the *system state space*. We illustrate below the system state on our two examples.

Admission control $\mathbf{x} = (\underbrace{x}_{\mathbf{s_1}}, \underbrace{\mu, \lambda_1, \cdots, \lambda_n, h, c_1, \cdots, c_n, \eta}_{\mathbf{s_2}})$.

Tandem queue $\mathbf{x} = (\underbrace{x_1, x_2}_{\mathbf{s_1}}, \underbrace{\mu_1, \mu_2, \lambda, h_1, h_2, b, \eta}_{\mathbf{s_2}})$.

Let $v^*(\mathbf{x}) = v^*(\mathbf{s_1}, \mathbf{s_2})$ be the optimal expected discounted cost over an infinite horizon when the initial state is $\mathbf{s_1}$ and system parameters are given by $\mathbf{s_2}$.

$$v^\star(\mathbf{x}) = \boldsymbol{\mathcal{O}} v^\star(\mathbf{x}), \tag{1}$$

where $\boldsymbol{\mathcal{O}}$ is the optimal operator.

We assume that the optimal operator can be decomposed as a convex combination of operators, using the well-known method of uniformization (Lippman 1975):

$$\boldsymbol{\mathcal{O}} v(\mathbf{x}) = \frac{1}{\eta + \sum_{i=0}^{l} p_i} \left(\mathcal{H}(\mathbf{x}) + \sum_{i=1}^{l} p_i \mathcal{O}_i(\mathbf{x}) v + p_0 v(\mathbf{x}) \right). \tag{2}$$

The operator $\mathcal{H}$ is a cost rate function which does not depend on decisions. The operator $\mathcal{O}_i$ is associated to the i-th type of event which occurs with rate p_i. The last term $p_0 v$ corresponds to a fictitious event which occurs with rate p_0 and affects neither the state nor the cost of the system. This term will be useful to compare systems with different event rates or discount rates in order to keep constant the quantity $\eta + \sum_{i=0}^{l} p_i$. For instance, if the arrival rate increases of ϵ, the fictitious rate decreases of ϵ. Without loss of generality, we set $\eta + \sum_{i=0}^{l} p_i = 1$ which is equivalent to set a time unit.

In this paper, we consider two new operators that generalize several operators from the literature. Let $\mathbf{y}=\mathbf{x}+\mathbf{b}$. The *translation operator* $\mathcal{T}$ and the *choice operator* $\mathcal{C}$ are defined as

$$\mathcal{T}v(\mathbf{x})=\begin{cases} v(\mathbf{y}+\mathbf{a})+c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X},\\ v(\mathbf{y})+c_r & \text{otherwise.}\end{cases}$$

$$\mathcal{C}v(\mathbf{x})=\begin{cases} \min\{v(\mathbf{y}+\mathbf{a})+c_a, v(\mathbf{y})+c_b\} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X},\\ v(\mathbf{y})+c_r, & \text{otherwise.}\end{cases}$$

In the definition of $\mathcal{T}$ and $\mathcal{C}$, we implicitly assume that if $\mathbf{x}\in\mathcal{X}$ then $\mathbf{y}\in\mathcal{X}$. The decision in the choice operator depends on the sign of the cost difference $c_d=c_a-c_b$. Operator $\mathcal{C}$ will reduce to operator $\mathcal{T}$ if the optimal decision is always to move to state $\mathbf{y}+\mathbf{a}$ (for instance if c_b goes to infinity).

Table 2 illustrates how several operators from the literature (Koole 1998, 2006, Çil et al. 2009) can be seen as special cases of the translation and choice operators. In this table and the rest of the paper, $\mathbf{e}_i=(0,\dots,0,1,0,\dots,0)$ is the unit vector in direction i (the "1" is in i^{th} position).

We define two last operators $\Delta_{\boldsymbol{\alpha}}$ and $\Omega_{\mathcal{O}}$ with $\boldsymbol{\alpha}$ a translation of the system state and $\mathcal{O}$ an ad-hoc operator:

$$\Delta_{\boldsymbol{\alpha}}v(\mathbf{x})=v(\mathbf{x}+\boldsymbol{\alpha})-v(\mathbf{x}),$$
$$\Omega_{\mathcal{O}}v(\mathbf{x})=\mathcal{O}v(\mathbf{x})-v(\mathbf{x}).$$

The quantity $\Omega_{\mathcal{O}}v(\mathbf{x})$ represents the marginal cost associated to the decision made by operator $\mathcal{O}$.

4. Value function and state space properties

In the following definitions, $v\geq 0$ means that for all $\mathbf{s_1}$, $v(\mathbf{s_1},\mathbf{s_2})\geq 0$ (the value of $\mathbf{s_2}$ will be clear from the context). The word increasing (resp. decreasing, positive, negative) is used for non-decreasing (resp. non-increasing, non-negative, non-positive).

We first define some properties of a value function:

$$\texttt{P}: v\geq 0 \quad \text{(positive)}\ ,$$
$$\texttt{N}: v\leq 0 \quad \text{(negative)}\ ,$$
$$\texttt{I}_{\boldsymbol{\alpha}}: \Delta_{\boldsymbol{\alpha}}v\geq 0 \quad \text{(increasing)}\ ,$$
$$\texttt{D}_{\boldsymbol{\alpha}}: \Delta_{\boldsymbol{\alpha}}v\leq 0 \quad \text{(decreasing)}\ ,$$
$$\texttt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}: \Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}v\geq 0 \quad \text{(supermodularity)}\ ,$$
$$\texttt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}: \Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}v\leq 0 \quad \text{(submodularity)}\ .$$

We can see $\texttt{P}$, $\texttt{N}$, $\texttt{I}_{\boldsymbol{\alpha}}$, $\texttt{D}_{\boldsymbol{\alpha}}$, $\texttt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\texttt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ as Boolean variables. For instance, $\texttt{I}_{\boldsymbol{\alpha}}$ is true if the assertion "$\Delta_{\boldsymbol{\alpha}}v\geq 0$" is true. We will use notation $\wedge$ (resp. $\vee$) for Boolean operator "and" (resp. "or").

Table 2 Some operators from the literature (Koole 1998, 2006, Çil et al. 2009) as special cases of the translation and choice operators. Unless specified, we set $\mathbf{b}=\mathbf{0}$, $c_a=c_b=c_r=c_a=0$ and $\mathcal{S}_1=(\mathbb{Z}^+)^{m_1}$

Name	Operator from the literature	With choice and translation operators
Arrival	$T_{A(i)}v(\mathbf{x})=v(\mathbf{x}+\mathbf{e}_i)$	$\mathcal{T}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_i$
Controlled arrival	$T_{CA(i)}v(\mathbf{x})=\min\{v(\mathbf{x});v(\mathbf{x}+\mathbf{e}_i)+c\}$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_i,\ c_a=c$
Controlled arrival as fork	$T_{CAF}v(\mathbf{x})=\min\{v(\mathbf{x});v(\mathbf{x}+\sum_i\mathbf{e}_i)+c\}$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=\sum_i\mathbf{e}_i,\ c_a=c$
Routing	$T_{R(i,j)}v(\mathbf{x})=\min_{k\in\{i,j\}}v(\mathbf{x}+\mathbf{e}_k)+c^k$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_j-\mathbf{e}_i$, $\mathbf{b}=\mathbf{e}_i,\ c_a=c^j,\ c_b=c^i$
Batch arrival	$T_{BA(i)}v(\mathbf{x})=\min_{0\geq j\geq B}v(\mathbf{x}+j\mathbf{e}_i)+j$	$\mathcal{C}_1(\mathcal{C}_2(\dots(\mathcal{C}_Bv)\dots)(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_i,\ c_a=c$
Departure	$T_{D(i)}v(\mathbf{x})=v((\mathbf{x}-\mathbf{e}_i)^+)$	$\mathcal{T}v(\mathbf{x})$ with $\mathbf{a}=-\mathbf{e}_i$
Controlled departure	$T_{CD(i)}v(\mathbf{x})=\begin{cases}\min\{v(\mathbf{x}),v(\mathbf{x}-\mathbf{e}_i)+c\} \text{ if } x_i>0,\\ v(\mathbf{x}) \text{ else,}\end{cases}$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=-\mathbf{e}_i,\ c_a=c$
Parallel departure	$T_{PD}v(\mathbf{x})=\sum_{i=1}^l\gamma_iv((\mathbf{x}-\mathbf{e}_i)^+)$	$\sum_{i=1}^l\gamma_i\mathcal{T}_iv(\mathbf{x})$ with $\mathbf{a}_i=-\mathbf{e}_i$
Tandem server	$T_{T(i,j)}v(\mathbf{x})=v((\mathbf{x}-\mathbf{e}_i+\mathbf{e}_j)^+)$	$\mathcal{T}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_j-\mathbf{e}_i$
Controlled tandem server	$T_{CT(i,j)}v(\mathbf{x})=\begin{cases}\min\{v(\mathbf{x}),v(\mathbf{x}-\mathbf{e}_i+\mathbf{e}_j)+c\}\\ \quad \text{if } x_i>0,\\ v(\mathbf{x}) \text{ else.}\end{cases}$	$\mathcal{C}v(\mathbf{x})$ with $\mathbf{a}=\mathbf{e}_j-\mathbf{e}_i,\ c_a=c$

Moreover $|a|$ will be a Boolean variable which is true when the assertion "a" is *true*. Thus $\mathtt{I}_{\boldsymbol{\alpha}}=|\Delta_{\boldsymbol{\alpha}}v\geq 0|$. By convention, the "and" operator $\wedge$ takes precedence over the "or" operator $\vee$.

We show in A.1 that :

i) $\mathtt{I}_{\boldsymbol{\alpha}}=\mathtt{D}_{-\boldsymbol{\alpha}}$,

ii) $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}=\mathtt{S}^{ub}_{-\boldsymbol{\alpha},\boldsymbol{\beta}}=\mathtt{S}^{ub}_{\boldsymbol{\alpha},-\boldsymbol{\beta}}=\mathtt{S}_{-\boldsymbol{\alpha},-\boldsymbol{\beta}}$,

iii) $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\wedge\mathtt{S}_{\boldsymbol{\gamma},\boldsymbol{\beta}}$ implies $\mathtt{S}_{\boldsymbol{\alpha}+\boldsymbol{\gamma},\boldsymbol{\beta}}$,

For instance, property iii) means that if v is $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\mathtt{S}_{\boldsymbol{\gamma},\boldsymbol{\beta}}$, then v is $\mathtt{S}_{\boldsymbol{\alpha}+\boldsymbol{\gamma},\boldsymbol{\beta}}$.

We also define some properties related to the marginal cost operator $\Omega_{\mathcal{O}}$:

$$\mathtt{PM}(\mathcal{O}):\Omega_{\mathcal{O}}v\geq 0\ \text{ (positive marginal cost) },$$

$$\mathtt{NM}(\mathcal{O}):\Omega_{\mathcal{O}}v\leq 0\ \text{ (negative marginal cost) },$$

$$\mathtt{IM}_{\boldsymbol{\alpha}}(\mathcal{O}):\Delta_{\boldsymbol{\alpha}}\Omega_{\mathcal{O}}v\geq 0\ \text{ (increasing marginal cost) },$$

$$\mathtt{DM}_{\boldsymbol{\alpha}}(\mathcal{O}):\Delta_{\boldsymbol{\alpha}}\Omega_{\mathcal{O}}v\leq 0\ \text{ (decreasing marginal cost) }.$$

Again we can see $\mathtt{PM}(\mathcal{O})$, $\mathtt{NM}(\mathcal{O})$, $\mathtt{IM}_{\boldsymbol{\alpha}}(\mathcal{O})$ and $\mathtt{DM}_{\boldsymbol{\alpha}}(\mathcal{O})$ as Boolean variables. We have $\mathtt{IM}_{\boldsymbol{\alpha}}(\mathcal{O}) = \mathtt{DM}_{-\boldsymbol{\alpha}}(\mathcal{O})$.

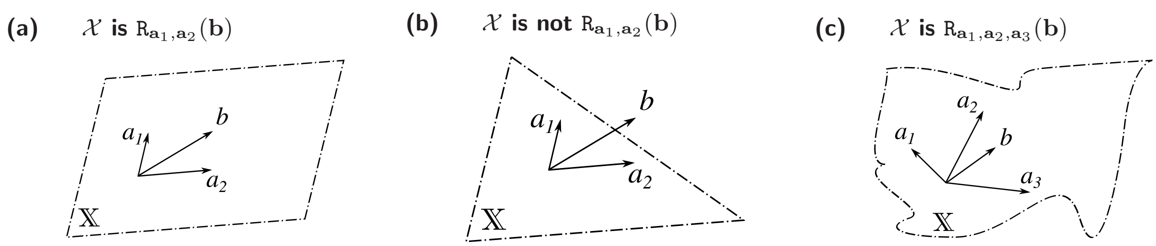
The form of the state space is also important when characterizing the optimal policy. The simplest case is when the state space is infinite in all directions. However results can be derived for other state spaces that can be described by the following property, illustrated in Figure 5.

Definition 1. A state space $\mathcal{X}$ is $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})$ if for all $\mathbf{x}$ such that $\{\mathbf{x}, \mathbf{x}+\mathbf{a}_1, \ldots, \mathbf{x}+\mathbf{a}_l\} \subset \mathcal{X}$, then $\mathbf{x}+\mathbf{b} \in \mathcal{X}$.

Again, we can see $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})$ as a Boolean variable which is true when $\mathcal{X}$ is $\mathtt{R}_{\mathbf{a}_1,\ldots}(\mathbf{b})$. One can easily check the following properties (see A.2 for a proof for vi) :

i) $\mathtt{R}(\mathbf{b})$: the set $\mathcal{X}$ is invariant by translation $\mathbf{b}$
ii) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{0}) = true$
iii) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{a}_i) = true$
iv) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_i,\mathbf{a}_j,\ldots,\mathbf{a}_l}(\mathbf{b}) = \mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_j,\mathbf{a}_i,\ldots,\mathbf{a}_l}(\mathbf{b})$
v) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_{l-1}}(\mathbf{b})$ implies $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b})$
vi) $\mathtt{R}_{\mathbf{a}_1,\ldots,\mathbf{a}_l}(\mathbf{b}) = \mathtt{R}_{\mathbf{a}_1-\mathbf{a}_l,\ldots,\mathbf{a}_{l-1}-\mathbf{a}_l,-\mathbf{a}_l}(\mathbf{b}-\mathbf{a}_l)$

Figure 5 **Illustration of $\mathtt{R}$ properties on different system state spaces**



5. Properties of the operators

A system parameter perturbation $\boldsymbol{\epsilon} = (0, \ldots, 0, \epsilon_1, \ldots, \epsilon_{m_2})$, with $\epsilon_i \in \mathbb{R}$, translates the system state from $\mathbf{x}$ to $\mathbf{x} + \boldsymbol{\epsilon}$. Such a translation can modify the transition rates p_i, the discount rate η and the costs c_a, c_r, c_b of the operators. However a perturbation $\boldsymbol{\epsilon}$ does not change the state, the state space or the action space. Let $\epsilon_{p_i}, \epsilon_\eta, \epsilon_{c_a}, \epsilon_{c_r}, \epsilon_{c_b}$ denote the perturbation for parameters p_i, η, c_a, c_r, c_b.

In this section, we provide sufficient conditions such that the optimal value function is positive/negative, increasing/decreasing in the direction $\boldsymbol{\epsilon}$, convex/concave in the direction $\boldsymbol{\epsilon}$. We also provide sufficient conditions such that the optimal switching curves, if any, increase or decrease with $\boldsymbol{\epsilon}$.

Table 3 summarize our results for the operators. A detailed proof of each result can be found in B for the translation operator and in C for the choice operator. Section 6 will illustrate how to use these results for the admission control problem and the tandem queue problem.

5.1. Sign of the optimal cost

To study the effect of the discount rate on the optimal value function, we will need results on the sign of the optimal value function. This section is trivial but has the merit to introduce the approach and notations in a simple way.

The optimal value function $v^\star$ is positive (resp. negative) if the optimal operator $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{P}$ (resp. $\mathtt{N}$) (i.e. if v is $\mathtt{P}$ then $\boldsymbol{\mathcal{O}}v$ is $\mathtt{P}$). From (2), we have

Proposition 1. *$\boldsymbol{\mathcal{O}}$ propagates $\mathtt{P}$ if the following Boolean variable is true.*

$$|\mathcal{H} \geq 0| \bigwedge_{i=1}^{l} |\mathcal{O}_i \text{ propagate } \mathtt{P}|$$

$\boldsymbol{\mathcal{O}}$ propagates $\mathtt{N}$ if the following Boolean variable is true.

$$|\mathcal{H} \leq 0| \bigwedge_{i=1}^{l} |\mathcal{O}_i \text{ propagate } \mathtt{N}|$$

In order to apply Proposition 1, we need to prove that each individual operator $\mathcal{O}_i$ propagates $\mathtt{P}$ (or $\mathtt{N}$). Table 3 (cells 21 to 24) provide sufficient conditions for the translation and the choice operators to propagate $\mathtt{P}$ (or $\mathtt{N}$). These results are trivial and simply state that an operator propagates $\mathtt{P}$ (or $\mathtt{N}$) if all its costs are positive (or negative). The proof for each cell is in B and C.

We point out that the translation operator is a special case of choice operator when c_b tends to infinity (i.e. $\Delta_{\mathbf{a}} v + c_a - c_b \leq 0$). However, we kept the two operators to facilitate the model and the use of the results. It should be noted that our results for choice operator are sufficient conditions, and that we simplify the results assuming that $\Delta_{\mathbf{a}} v + c_a - c_b$ sometimes positive and sometimes negative (i.e. $\exists \mathbf{x}_1, \mathbf{x}_2$ such that $\Delta_{\mathbf{a}} v(\mathbf{x}_1) + \geq c_b - c_a$ and $\Delta_{\mathbf{a}} v(\mathbf{x}_2) \leq c_b - c_a$). That is why the results on left column is not a particular case of the right column when c_b tends to infinity. It is up to the user of our results to consider translation operator if $\Delta_{\mathbf{a}} v + c_a - c_b$ always positive or negative.

5.2. Monotonicity of the optimal cost

To study the monotonicity of the optimal value function in direction $\boldsymbol{\epsilon}$, we can limit our analysis to $\mathtt{I}_{\boldsymbol{\epsilon}}$ as $\mathtt{D}_{\boldsymbol{\epsilon}} = \mathtt{I}_{-\boldsymbol{\epsilon}}$. The optimal value function $v^\star$ is $\mathtt{I}_{\boldsymbol{\epsilon}}$ if the optimal operator $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$.

From (2), we have

$$\boldsymbol{\mathcal{O}}v(\mathbf{x}+\boldsymbol{\epsilon}) = \mathcal{H}(\mathbf{x}+\boldsymbol{\epsilon}) + \textstyle\sum_{i=1}^{l}(p_i + \epsilon_{p_i})\mathcal{O}_i v(\mathbf{x}+\boldsymbol{\epsilon}) + (p_0 - \epsilon_\eta - \sum_{i=1}^{l}\epsilon_{p_i})v(\mathbf{x}+\boldsymbol{\epsilon}). \tag{3}$$

From (2) and (3), it follows that

$$\Delta_{\boldsymbol{\epsilon}}\boldsymbol{\mathcal{O}}v(\mathbf{x}) = \begin{pmatrix} \Delta_{\boldsymbol{\epsilon}}\mathcal{H}(\mathbf{x}) \\ +p_0\Delta_{\boldsymbol{\epsilon}}v(\mathbf{x}) \\ +\sum_{i=1}^{l} p_i \Delta_{\boldsymbol{\epsilon}}\mathcal{O}_i v(\mathbf{x}) \\ +\sum_{i=1}^{l}\epsilon_{p_i}\Omega_{\mathcal{O}_i} v(\mathbf{x}+\boldsymbol{\epsilon}) \\ -\epsilon_\eta v(\mathbf{x}+\boldsymbol{\epsilon}) \end{pmatrix}. \tag{4}$$

Table 3 Sufficient conditions for properties of the operators

	Translation operator ($\mathcal{O}=\mathcal{T}$)	Choice operator ($\mathcal{O}=\mathcal{C}$)
$\mathcal{O}$ propagates $\mathtt{P}$	21) $\lvert c_a \geq 0\rvert \wedge \begin{pmatrix} \lvert c_r \geq 0\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	22) $\lvert c_a \geq 0\rvert \wedge \lvert c_b \geq 0\rvert \wedge \begin{pmatrix} \lvert c_r \geq 0\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
$\mathcal{O}$ propagates $\mathtt{N}$	23) $\lvert c_a \leq 0\rvert \wedge \begin{pmatrix} \lvert c_r \leq 0\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	24) $\lvert c_a \leq 0\rvert \wedge \lvert c_b \leq 0\rvert \wedge \begin{pmatrix} \lvert c_r \leq 0\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
$\mathcal{O}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$	25) $\lvert \epsilon_{c_a} \geq 0\rvert \wedge \begin{pmatrix} \lvert \epsilon_{c_r} \geq 0\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	26) $\lvert \epsilon_{c_a} \geq 0\rvert \wedge \lvert \epsilon_{c_b} \geq 0\rvert \wedge \begin{pmatrix} \lvert \epsilon_{c_r} \geq 0\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
$\mathcal{O}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$	27) *true*	28) $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \lvert \epsilon_{c_d} = 0\rvert$
$\mathcal{O}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$	29) *true*	30) $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \lvert \epsilon_{c_d} \geq 0\rvert \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge \lvert \epsilon_{c_d} \leq 0\rvert$
$\mathcal{O}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$	31) $\begin{pmatrix} \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \lvert \epsilon_{c_r} \geq \epsilon_{c_a}\rvert \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \lvert \epsilon_{c_a} \geq \epsilon_{c_r}\rvert \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$	32) $\begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \lvert \epsilon_{c_d} \leq 0\rvert \\ \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \lvert \epsilon_{c_d} \geq 0\rvert \\ \vee \mathtt{S}_{\mathbf{d}+\mathbf{a},\boldsymbol{\epsilon}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \vee \mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}) \wedge \lvert \epsilon_{c_d} = 0\rvert \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d}-\mathbf{a},\boldsymbol{\epsilon}} \wedge \lvert \epsilon_{c_r} \geq \epsilon_{c_a}\rvert \wedge \lvert \epsilon_{c_r} \geq \epsilon_{c_b}\rvert \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}+\mathbf{a}} \wedge \lvert \epsilon_{c_a} \geq \epsilon_{c_r}\rvert \wedge \lvert \epsilon_{c_b} \geq \epsilon_{c_r}\rvert \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$
Positive marginal cost: $\Omega_{\mathcal{O}} v \geq 0$	33) $\lvert \Delta_{\mathbf{a}+\mathbf{b}} v \geq -c_a\rvert \wedge \begin{pmatrix} \lvert \Delta_{\mathbf{b}} v \geq -c_r\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	34) $\lvert \Delta_{\mathbf{b}} v \geq -c_b\rvert \wedge \lvert \Delta_{\mathbf{a}+\mathbf{b}} v \geq -c_a\rvert \wedge \begin{pmatrix} \lvert \Delta_{\mathbf{b}} v \geq -c_r\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
Negative marginal cost: $\Omega_{\mathcal{O}} v \leq 0$	35) $\lvert \Delta_{\mathbf{a}+\mathbf{b}} v \leq -c_a\rvert \wedge \begin{pmatrix} \lvert \Delta_{\mathbf{b}} v \leq -c_r\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$	36) $\begin{pmatrix} \lvert \Delta_{\mathbf{b}} v \leq -c_b\rvert \\ \vee \lvert \Delta_{\mathbf{a}+\mathbf{b}} v \leq -c_a\rvert \end{pmatrix} \wedge \begin{pmatrix} \lvert \Delta_{\mathbf{b}} v \leq -c_r\rvert \\ \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \end{pmatrix}$
Increasing marginal cost : $\Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}} v \geq 0$	37) $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}+\mathbf{b}} \wedge \lvert \epsilon_{c_a} \geq 0\rvert \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \lvert \epsilon_{c_r} \geq 0\rvert \\ \vee \mathtt{R}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$	38) $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{a}} \wedge \lvert \epsilon_{c_a} \geq 0\rvert \wedge \lvert \epsilon_{c_b} \geq 0\rvert \wedge \begin{pmatrix} \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{b}} \wedge \lvert \epsilon_{c_r} \geq 0\rvert \\ \vee \mathtt{R}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$
Increasing marginal cost: $\Delta_{\mathbf{d}} \Omega_{\mathcal{O}} v \geq 0$	39) $\mathtt{S}_{\mathbf{d},\mathbf{a}+\mathbf{b}} \wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{b}} \\ \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \\ \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix} \wedge \begin{pmatrix} \lvert \Delta_{\mathbf{a}} v \leq c_r - c_a\rvert \wedge [\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}}] \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \lvert \Delta_{\mathbf{a}} v \geq c_r - c_a\rvert \wedge [\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}] \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$	40) $\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \begin{pmatrix} \lvert c_r \geq \max\{-c_b, c_b\}\rvert \\ \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} \wedge \lvert \Delta_{\mathbf{a}} v \leq c_r - c_a\rvert \wedge \lvert c_r \geq c_b\rvert \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \lvert c_b \geq c_r\rvert \wedge \lvert \Delta_{\mathbf{a}} v \geq c_r - c_a\rvert \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$

This quantity is positive if each line is positive. The sign of the first line depends on the problem under consideration. The second line is positive if v is $\mathtt{I}_{\boldsymbol{\epsilon}}$. As $p_i > 0$, the third line is positive if each operator $\mathcal{O}_i$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$. The fourth line is positive if ϵ_{p_i} and the marginal cost $\Omega_{\mathcal{O}_i} v$ have the same sign, or if $\epsilon_{p_i} = 0$. Finally the last line is positive if v and ϵ_η have opposite signs, or if $\epsilon_\eta = 0$.

Using Boolean notations, this leads to the following proposition which provides sufficient conditions for the optimal operator to propagate $\mathtt{I}_{\boldsymbol{\epsilon}}$.

Proposition 2. *$\boldsymbol{\mathcal{O}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if the following Boolean variable is true.*

$$|\Delta_{\boldsymbol{\epsilon}}\mathcal{H} \geq 0| \bigwedge_{i=1}^{l} \left[\begin{array}{c} |\mathcal{O}_i \textit{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_{p_i} < 0| \wedge |\Omega_{\mathcal{O}_i} v \leq 0| \\ \vee |\epsilon_{p_i} > 0| \wedge |\Omega_{\mathcal{O}_i} v \geq 0| \\ \vee |\epsilon_{p_i} = 0| \end{pmatrix} \end{array} \right] \bigwedge \begin{pmatrix} |\epsilon_\eta < 0| \wedge |v \textit{ is } \mathtt{P}| \\ \vee |\epsilon_\eta > 0| \wedge |v \textit{ is } \mathtt{N}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}.$$

To apply Proposition 2, we need to prove that each individual operator $\mathcal{O}_i$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ (see cells 25 and 26 of Table 3 for sufficient conditions). For each operator $\mathcal{O}_i$ such that $\epsilon_{p_i} \neq 0$, we also need to show that the marginal cost is either positive or negative (see cells 33 and 34 for sufficient conditions).

The formalism used in Proposition 2 and Table 3 represent in a compact way the effect of many parameters for a large class of operators. When considering the effect of a single parameter Proposition 2 and Table 3 reduce drastically. For example consider the effect of an increase $\epsilon_{p_1} > 0$ of parameter p_1 on the optimal value function. From Proposition 2, $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{I}_{\epsilon_{p_1}}$ if each $\mathcal{O}_i$ propagates $\mathtt{I}_{\epsilon_{p_1}}$ and if $\Omega_{\mathcal{O}_1} v \geq 0$. From cells 25 and 26, the translation and choice operators propagates $\mathtt{I}_{\epsilon_{p_1}}$ since $\epsilon_{c_a} = \epsilon_{c_r} = \epsilon_{c_b} = 0$. Remains to check the positivity of the marginal cost.

5.3. Convexity or concavity of the optimal cost

A value function v is convex in direction $\boldsymbol{\epsilon}$ if it is $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$, i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}) \geq 0$. It is concave in direction $\boldsymbol{\epsilon}$ if it is $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$.

We want to find sufficient conditions such that operator $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ or $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. From (2), we have

$$\begin{aligned} \boldsymbol{\mathcal{O}} v(\mathbf{x}+2\boldsymbol{\epsilon}) &= \mathcal{H}(\mathbf{x}+2\boldsymbol{\epsilon}) + \textstyle\sum_{i=1}^{l} (p_i + 2\epsilon_{p_i}) \mathcal{O}_i v(\mathbf{x}+2\boldsymbol{\epsilon}) \\ &\quad + (p_0 - 2\epsilon_\eta - 2\textstyle\sum_{i=1}^{l} \epsilon_{p_i}) v(\mathbf{x}+2\boldsymbol{\epsilon}). \end{aligned} \tag{5}$$

From (2), (3) and (5), it follows that

$$\begin{aligned} \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \boldsymbol{\mathcal{O}} v(\mathbf{x}) &= \boldsymbol{\mathcal{O}} v(\mathbf{x}+2\boldsymbol{\epsilon}) - 2\boldsymbol{\mathcal{O}} v(\mathbf{x}+\boldsymbol{\epsilon}) + \boldsymbol{\mathcal{O}} v(\mathbf{x}) \\ &= \begin{pmatrix} \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \mathcal{H}(\mathbf{x}) \\ + p_0 \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}) \\ + \sum_{i=1}^{l} p_i \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}} \mathcal{O}_i v(\mathbf{x}) \\ + 2\sum_{i=1}^{l} \epsilon_{p_i} \Delta_{\boldsymbol{\epsilon}} \Omega_{\mathcal{O}_i} v(\mathbf{x}+\boldsymbol{\epsilon}) \\ - 2\epsilon_\eta \Delta_{\boldsymbol{\epsilon}} v(\mathbf{x}+\boldsymbol{\epsilon}) \end{pmatrix}. \end{aligned}$$

This quantity is positive (respectively negative) if each line is positive (respectively negative). This leads to the following proposition.

Proposition 3. *$\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ if the following Boolean variable is true.*

$$|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{H}\geq 0|\bigwedge_{i=1}^{l}\left[\begin{array}{c}|\mathcal{O}_i \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}|\\ \wedge\left(\begin{array}{c}|\epsilon_{p_i}<0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i}v\leq 0|\\ \vee|\epsilon_{p_i}>0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i}v\geq 0|\\ \vee|\epsilon_{p_i}=0|\end{array}\right)\end{array}\right]\bigwedge\left(\begin{array}{c}|\epsilon_\eta<0|\wedge|v \text{ is } \mathtt{I}_{\boldsymbol{\epsilon}}|\\ \vee|\epsilon_\eta>0|\wedge|v \text{ is } \mathtt{I}_{-\boldsymbol{\epsilon}}|\\ \vee|\epsilon_\eta=0|\end{array}\right).$$

$\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ if the following Boolean variable is true.

$$|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{H}\leq 0|\bigwedge_{i=1}^{l}\left[\begin{array}{c}|\mathcal{O}_i \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}|\\ \wedge\left(\begin{array}{c}|\epsilon_{p_i}>0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i}v\leq 0|\\ \vee|\epsilon_{p_i}<0|\wedge|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_i}v\geq 0|\\ \vee|\epsilon_{p_i}=0|\end{array}\right)\end{array}\right]\bigwedge\left(\begin{array}{c}|\epsilon_\eta>0|\wedge|v \text{ is } \mathtt{I}_{\boldsymbol{\epsilon}}|\\ \vee|\epsilon_\eta<0|\wedge|v \text{ is } \mathtt{I}_{-\boldsymbol{\epsilon}}|\\ \vee|\epsilon_\eta=0|\end{array}\right).$$

In order to apply Proposition 3, we need to prove that each individual operator $\mathcal{O}_i$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ or $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ (see cells 27 to 30 for sufficient conditions). For each operator $\mathcal{O}_i$ such that $\epsilon_{p_i}\neq 0$, we also need to show that the marginal cost is either increasing or decreasing in the direction $\boldsymbol{\epsilon}$ (see cells 37 to 38 for sufficient conditions).

5.4. Monotonicity of the optimal policy

In this section, we study the effect of a system parameter perturbation $\boldsymbol{\epsilon}$ on the optimal policy. For the choice operator $\mathcal{C}$, the decision depends on the sign of $\Delta_{\mathbf{a}}v(\mathbf{x})+c_d$. If $\Delta_{\mathbf{a}}v(\mathbf{x})+c_d\geq 0$, it is optimal to stay in state $\mathbf{x}$. Otherwise, it is optimal to go in state $\mathbf{x}+\mathbf{a}$. The sign of $\Delta_{\boldsymbol{\epsilon}}(\Delta_{\mathbf{a}}v+c_d)=\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}}v+\epsilon_{c_d}$ will provide an indication on how the optimal policy evolves with $\boldsymbol{\epsilon}$.

Our objective is to find sufficient conditions to have $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}}v+\epsilon_{c_d}$ positive. If $\epsilon_{c_d}\geq 0$, it is sufficient to show that $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}}v\geq 0$ (i.e. v is $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}}$). When ϵ_{c_d} is negative, we can not conclude. Sufficient conditions to have $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{a}}v+\epsilon_{c_d}$ negative can be easily deduced by noticing that $\mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}}=\mathtt{S}_{\mathbf{a},-\boldsymbol{\epsilon}}$.

Let $\mathbf{d}$ be a vector in $\mathcal{X}$ that translates the state but does not change the system parameters. From (4), we have

$$\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\boldsymbol{\mathcal{O}}v(\mathbf{x})=\left(\begin{array}{c}\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\mathcal{H}(\mathbf{x})\\ +p_0\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{x})\\ +\sum_{i=1}^{l}p_i\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\mathcal{O}_iv(\mathbf{x})\\ +\sum_{i=1}^{l}\epsilon_{p_i}\Delta_{\mathbf{d}}\Omega_{\mathcal{O}_i}v(\mathbf{x}+\boldsymbol{\epsilon})\\ -\epsilon_\eta\Delta_{\mathbf{d}}v(\mathbf{x}+\boldsymbol{\epsilon})\end{array}\right).$$

This quantity is positive if each line is positive . This leads to the following proposition.

Proposition 4. *$\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ if the following Boolean variable is true.*

$$|\Delta_{\mathbf{d}}\Delta_{\boldsymbol{\epsilon}}\mathcal{H}\geq 0|\bigwedge_{i=1}^{l}\left[\begin{array}{c}|\mathcal{O}_i \text{ propagates } \mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}|\\ \wedge\left(\begin{array}{c}|\epsilon_{p_i}<0|\wedge|\Delta_{\mathbf{d}}\Omega_{\mathcal{O}_i}v\leq 0|\\ \vee|\epsilon_{p_i}>0|\wedge|\Delta_{\mathbf{d}}\Omega_{\mathcal{O}_i}v\geq 0|\\ \vee|\epsilon_{p_i}=0|\end{array}\right)\end{array}\right]\bigwedge\left(\begin{array}{c}|\epsilon_\eta<0|\wedge|v \text{ is } \mathtt{I}_{\mathbf{d}}|\\ \vee|\epsilon_\eta>0|\wedge|v \text{ is } \mathtt{I}_{-\mathbf{d}}|\\ \vee|\epsilon_\eta=0|\end{array}\right).$$

In order to apply Proposition 4, we need to prove that each individual operator $\mathcal{O}_i$ propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (see cells 31 and 32 for sufficient conditions). For each operator $\mathcal{O}_i$ such that $\epsilon_{p_i}\neq 0$, we also need to show that the marginal cost is either increasing or decreasing in the direction $\mathbf{d}$ (see cells 39 and 40 for sufficient conditions).

6. Illustration of results

We illustrate in this section how to apply the results of Table 3 to the admission control problem and the tandem queue problem.

6.1. Admission control problem

The optimality equations for the admission control introduced in Section 1 are

$$\begin{aligned}
\boldsymbol{\mathcal{O}}v &= \mathcal{H} + \mu\mathcal{O}_0 v + \textstyle\sum_{i=1}^{n} \lambda_i \mathcal{O}_i v + p_0 v,\\
\mathcal{H}(\mathbf{x}) &= hx,\\
\mathcal{O}_0 v(\mathbf{x}) &= v[(\mathbf{x}-\mathbf{e}_1)^+] = \mathcal{T}v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = -\mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_a = c_r = 0, \end{cases}\\
\mathcal{O}_i v(\mathbf{x}) &= \min(v(\mathbf{x}) + c_i, v(\mathbf{x}+\mathbf{e}_1)) \text{ for } i = 1, \cdots, n = \mathcal{C}v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = \mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_b = c_i, c_a = c_r = 0. \end{cases}
\end{aligned}$$

The state space is $\mathcal{S}_1 = \mathbb{Z}^+$.

The optimal policy has been characterized in Stidham (1985). It consists of n thresholds $t_1, \cdots, t_n$. Customers of class i are accepted in the system if $x < t_i$ and rejected otherwise. If the rejection costs are ordered as $c_1 \geq \cdots \geq c_n$, then $t_1 \geq \cdots \geq t_n$. Finally the optimal value function is convex and increasing ($\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$ and $\mathtt{I}_{\mathbf{e}_1}$). Çil et al. (2009) have shown that the optimal thresholds t_i are increasing in the service rate μ and decreasing in the arrival rates λ_i.

Using propositions 1, 2, 3, 4 and Table 3, we re-obtain these results and complete them in several directions.

Theorem 1. *In the admission control problem, the optimal value function and the optimal cost have the following properties.*

- ***Monotonicity:*** *The optimal value function is increasing in the arrival rates λ_i, the rejection costs c_i, the holding cost h and decreasing in the service rate μ and the discount rate η.*
- ***Convexity/concavity:*** *The optimal value function is concave in the holding cost h.*
- ***Monotonicity of the optimal policy:*** *The optimal thresholds t_i are decreasing in the arrival rate λ_i, the holding cost h, and increasing in the service rate μ and the discount rate η.*

Each result of Theorem 1 is proven in D.1. To illustrate the methodology, we provide below a detailed proof for the effect of the lambda rate λ_1 on the optimal cost.

Assume that v is $\mathtt{I}_{\epsilon_{\lambda_1}}$. From Proposition 2, $\boldsymbol{\mathcal{O}}v$ is $\mathtt{I}_{\epsilon_{\lambda_1}}$ if

$$|\Omega_{\mathcal{O}_1} v \geq 0| \wedge |\bigwedge_{i=0}^{n} |\mathcal{O}_i \text{ propagates } \mathtt{I}_{\epsilon_{\lambda_1}}|.$$

is true. From Cell 34 of Table 3, $\Omega_{\mathcal{O}_1} v \geq 0$ if

$$|\Delta_{\mathbf{0}} v \geq -c_i| \wedge |\Delta_{\mathbf{e}_1} v \geq 0| \wedge \begin{pmatrix} |\Delta_{\mathbf{0}} v \geq 0| \\ \vee \mathtt{R}_{\mathbf{0}}(\mathbf{e}_1) \end{pmatrix} = |\Delta_{\mathbf{e}_1} v \geq 0|$$

is true. From Cell 25 of Table 3, $\mathcal{O}_0$ propagates $\mathtt{I}_{\epsilon_{\lambda_1}}$ without condition. From Cell 26 of Table 3, $\mathcal{O}_i$ propagates $\mathtt{I}_{\epsilon_{\lambda_1}}$ without condition, for $i=1,\cdots,n$. In the end, $\boldsymbol{\mathcal{O}}v$ is $\mathtt{I}_{\epsilon_{\lambda_1}}$ if v is $\mathtt{I}_{\epsilon_{\lambda_1}}$ and $\mathtt{I}_{\mathbf{e}_1}$.

Assume that v is $\mathtt{I}_{\epsilon_{\lambda_1}}$ and $\mathtt{I}_{\mathbf{e}_1}$, then $\boldsymbol{\mathcal{O}}v$ is $\mathtt{I}_{\epsilon_{\lambda_1}}$ from the previous paragraph. Moreover $\boldsymbol{\mathcal{O}}v$ is $\mathtt{I}_{\mathbf{e}_1}$ from Stidham (1985). By value iteration, the optimal value function $v^\star$ is $\mathtt{I}_{\epsilon_{\lambda_1}}$ and $\mathtt{I}_{\mathbf{e}_1}$.

We can also derive piecewise results by looking at the effect of parameters for a set of fixed thresholds $t_1,\cdots,t_n$. If customers of class i are accepted if and only if $x_i<t_i$, operator $\mathcal{O}_i$ is replaced by the following operator that is a translation operator.

$$\tilde{\mathcal{O}}_i v(\mathbf{x}) = \begin{cases} v(\mathbf{x}+\mathbf{e}) \text{ if } s\in\{0,\dots,t_i-1\}, \\ v(\mathbf{x})+c_i \text{ otherwise,} \end{cases} = \mathcal{T}v(\mathbf{x}) \text{ with } \begin{cases} \mathcal{S}_1^i=\{0,\dots,t_i-1\}, \\ \mathbf{a}=\mathbf{e}_1, \mathbf{b}=\mathbf{0}, \\ c_a=c_r=0. \end{cases}$$

Using again the results of Table 3, we obtain the following theorem. The proof is in D.2.

Theorem 2. *The optimal value function is piecewise linear in the rejection costs c_i and the holding cost h and piecewise convex in the arrival rates λ_i and the service rate μ.*

This theorem is illustrated in Section 1 (see Figure 2).

6.2. Tandem queue problem

The optimality equations for the tandem queue problem are

$$\begin{aligned}
\boldsymbol{\mathcal{O}}v &= \mathcal{H}+\mu_1\mathcal{O}_1 v+\mu_2\mathcal{O}_2 v+\lambda\mathcal{O}_3 v+p_0 v,\\
\mathcal{H}(\mathbf{x}) &= h_1x_1+h_2\max\{x_2,0\}+b\max\{-x_2,0\},\\
\mathcal{O}_1 v(\mathbf{x}) &= \min(v(\mathbf{x}),v(\mathbf{x}+\mathbf{e}_1)) = \mathcal{C}v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a}=\mathbf{e}_1,\mathbf{b}=\mathbf{0}, \\ c_a=c_r=0, \end{cases}\\
\mathcal{O}_2 v(\mathbf{x}) &= \begin{cases} \min(v(\mathbf{x}),v(\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2)) & \text{if } x_1>0, \\ v(\mathbf{x}) & \text{else,} \end{cases} = \mathcal{C}v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a}=\mathbf{e}_2-\mathbf{e}_1,\mathbf{b}=\mathbf{0}, \\ c_a=c_b=c_r=0, \end{cases}\\
\mathcal{O}_3 v(\mathbf{x}) &= v(\mathbf{x}-\mathbf{e}_2) = \mathcal{T}v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a}=-\mathbf{e}_2,\mathbf{b}=\mathbf{0}, \\ c_a=c_b=c_r=0. \end{cases}
\end{aligned}$$

From Veatch and Wein (1992) the optimal policy consists of two switching curves: Produce at M_i iff $x_2<s_i(x_1)$, for $i=1,2$. Moreover the optimal value function is $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1-\mathbf{e}_2,\mathbf{e}_1}$, and $\mathtt{S}_{\mathbf{e}_2-\mathbf{e}_1,\mathbf{e}_2}$. Using propositions 1, 2, 3, 4 and Table 3, we obtain the following new results for this problem.

Theorem 3. *In the tandem queue problem, the optimal value function and the optimal cost have the following properties.*

- ***Monotonicity:*** *The optimal value function is increasing in the costs h_i and b, and decreasing in the service rate μ_i and the discount rate η.*
- ***Convexity/concavity:*** *The optimal value function is concave in the costs h_2 and b.*
- ***Monotonicity of the optimal policy:*** *the optimal switching curves $s_i(x_1)$ are increasing in the demand rate λ, the backlog costs b, and decreasing in the holding cost h_2.*

Each result of Theorem 3 is proven in E. This theorem is illustrated in Section 1 for the effect of λ (see Figure 4a).

7. Compensation

We focus this section on the admission control problem but the results we derive can be adapted to other situations. For the admission control problem, we have shown in Theorem 1 that the optimal cost increase with demand rates λ_1 and λ_2 and that the optimal thresholds t_1 and t_2 decrease in the demand rates λ_1 and λ_2. In this section, we would like to study e.g. the effect of a simultaneous decrease in λ_1 and increase in λ_2. Intuitively if $c_1 \leq c_2$, the optimal cost should increase if λ_1 decreases less than λ_2 increases. We call this phenomenon compensation between perturbations.

7.1. Effect on the optimal cost

In (4) we saw that $\sum_{i=1}^{n} \epsilon_{p_i} \Omega_{\mathcal{O}_i} v$ is positive if $\epsilon_{p_i} \geq 0$ and $\Omega_{\mathcal{O}_i} v \geq 0$ for all i. The following lemma provides a weaker condition.

LEMMA 1. *Consider two sequences of real numbers (u_i) and (v_i) and set $v_0 = 0$. We have*

$$\sum_{i=1}^{n} u_i v_i = \sum_{k=1}^{n} \left[\left(\sum_{i=k}^{n} u_i \right) (v_i - v_{i-1}) \right].$$

Moreover $\sum_{i=1}^{n} u_i v_i \geq 0$ if $\sum_{i=k}^{n} u_i \geq 0$ for all k and v_i is an increasing sequence.

From Table 3, the marginal cost $\Omega_{\mathcal{O}_i} v$ is positive and increasing c_i. Together with Lemma 1, it implies the following theorem.

THEOREM 4. *In the admission control problem, the optimal value function is increasing in $\boldsymbol{\epsilon}$ if $c_1 \leq \cdots \leq c_n$, $\sum_{i=k}^{n} \epsilon_{\lambda_i} \geq 0$, for $k = 1, \cdots, n$, $\epsilon_h \geq 0$, $\epsilon_\mu \leq 0$, $\epsilon_{c_i} \geq 0$, and $\eta \leq 0$.*

For two classes of customers, this theorem proves the intuitive behavior presented in the introduction of this section. It is also consistent with the example with three class of customers in Table 1.

7.2. Effect on the optimal policy

For the admission control problem, remind that $\mathcal{O}_i v(\mathbf{x}) = \min(v(\mathbf{x}) + c_i, v(\mathbf{x} + \mathbf{e}_1))$ for $i = 1, \cdots, n$. We can easily show that the marginal cost $\Omega_{\mathcal{O}_i} v(\mathbf{x}) = \min\{c_i, \Delta_{\mathbf{e}} v(\mathbf{x} + \mathbf{e}_1)\}$ is increasing in c_i. Together with Lemma 1, it implies the following theorem.

THEOREM 5. *In the admission control problem, the optimal thresholds t_i are decreasing in $\boldsymbol{\epsilon}$ if $c_1 \leq \cdots \leq c_n$, $\sum_{i=k}^{n} \epsilon_{\lambda_i} \geq 0$, for $k = 1, \cdots, n$, $\epsilon_h \geq 0$, $\epsilon_\mu \leq 0$, $\epsilon_{c_i} = 0$, and $\eta \leq 0$.*

8. Conclusion

This paper provides a general framework to study the effect of system parameters changes on the optimal cost and the optimal policy in multi-dimensional queueing control problem. We introduce two generic operators that cover many operators considered in the literature. For these operators, we derive sufficient conditions on the state space and the value function to guarantee the propagation of several properties of the value function and the marginal cost (sign, monotonicity supermodularity). We also show how to apply our results on two examples where we derive some new results.

Another contribution of the paper is to formalize a number of proofs that can be found in the literature and to investigate in a systematic way a set of necessary conditions through Boolean equations. We believe that our approach opens interesting perspectives on the automation of proofs of structural properties. A software tool would be particularly valuable for checking proofs from the literature and deriving new results that might be too complex to tackle manually.

Appendix A: Properties

A.1. Properties on the value function

i) and ii) Direct consequence of the definitions of $\mathtt{I}_{\boldsymbol{\alpha}}$, $\mathtt{D}_{\boldsymbol{\alpha}}$, $\mathtt{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\mathtt{S}^{ub}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

iii) We sum the two inequalities $\Delta_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\beta}}v(\mathbf{x}+\boldsymbol{\gamma})\geq 0$ and $\Delta_{\boldsymbol{\gamma}}\Delta_{\boldsymbol{\beta}}v\geq 0$ to get $\Delta_{\boldsymbol{\alpha}+\boldsymbol{\gamma}}\Delta_{\boldsymbol{\beta}}v(\mathbf{x})\geq 0$.

A.2. Properties on the system state space

i) to v) Trivial

vi) $\mathtt{R}_{\mathbf{a}_1,\dots,\mathbf{a}_l}(\mathbf{b})$ is equivalent to "for all $\mathbf{x}$ such that $\{\mathbf{x},\mathbf{x}+\mathbf{a}_1,\dots,\mathbf{x}+\mathbf{a}_l\}\subset\mathcal{X}$, $\mathbf{x}+\mathbf{b}\in\mathcal{X}$". In this assertion we replace $\mathbf{x}$ by $\mathbf{x}+\mathbf{a}_l$ to obtain "for all $\mathbf{x}$ such that $\{\mathbf{x}-\mathbf{a}_l,\mathbf{x}+\mathbf{a}_1-\mathbf{a}_l,\dots,\mathbf{x}\}\subset\mathcal{X}$, $\mathbf{x}+\mathbf{b}\in\mathcal{X}$'. So $\mathtt{R}_{\mathbf{a}_1,\dots,\mathbf{a}_l}(\mathbf{b})=\mathtt{R}_{-\mathbf{a}_l,\mathbf{a}_1-\mathbf{a}_l,\dots,\mathbf{a}_{l-1}-\mathbf{a}_l,\mathbf{0}}(\mathbf{b}-\mathbf{a}_l)$.

Appendix B: Translation operator

With $\mathbf{y}=\mathbf{x}+\mathbf{b}$ and $\forall\mathbf{x},\mathbf{x}+\mathbf{b}\in\mathcal{X}$,

$$\mathcal{T}v(\mathbf{x})=\begin{cases}v(\mathbf{y}+\mathbf{a})+c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ v(\mathbf{y})+c_r & \text{otherwise.}\end{cases}\tag{6}$$

B.1. Propagation of P and N (Cells 21 and 23)

We suppose that v is $\mathtt{P}$ (i.e. $v\geq 0$), then we want find conditions to have $\mathcal{T}$ which propagates $\mathtt{P}$ (i.e. $\mathcal{T}v\geq 0$). Given equation (6), we need to consider two cases:

- if $\mathbf{y}+\mathbf{a}\in\mathcal{X}$, then $\mathcal{T}v\geq 0$ if $c_a\geq 0$
- if $\mathbf{y}+\mathbf{a}\notin\mathcal{X}$, then $\mathcal{T}v\geq 0$ if $c_r\geq 0$. However this case is unreachable if $\mathcal{X}$ is $\mathtt{R}_{-\mathbf{b}}(\mathbf{a})$.

So $\mathcal{T}v\geq 0$ if $|c_a\geq 0|\wedge(\mathtt{R}_{-\mathbf{b}}(\mathbf{a})\vee|c_r\geq 0|)$. In the same way, $\mathcal{T}v\leq 0$ if $|c_a\leq 0|\wedge(\mathtt{R}_{-\mathbf{b}}(\mathbf{a})\vee|c_r\leq 0|)$.

B.2. Propagation of $\mathtt{I}_{\boldsymbol{\epsilon}}$ (Cell 25)

$$\Delta_{\boldsymbol{\epsilon}}\mathcal{T}v(\mathbf{x})=\begin{cases}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{y}+\mathbf{a})+\epsilon_{c_a} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\boldsymbol{\epsilon}}v(\mathbf{y})+\epsilon_{c_r} & \text{otherwise}\end{cases}$$

So $\mathcal{T}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if $|\epsilon_{c_a}\geq 0|\wedge(|\epsilon_{c_r}\geq 0|\vee\mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$

B.3. Propagation of $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (Cells 27 and 29)

We make the assumption that $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}v$ is positive (resp. negative), then we want find conditions to have $\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{T}$ positive (resp. negative).

$$\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}\mathcal{T}v(\mathbf{x})=\begin{cases}\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{y}+\mathbf{a}) & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}v(\mathbf{y}) & \text{otherwise}\end{cases}$$

So $\mathcal{T}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ or $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ without condition.

B.4. Propagation of $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (Cell 31)

We make the assumption that v is $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}v\geq 0$), then we want find conditions to have $\mathcal{T}$ which propagates $\mathtt{S}_{\mathbf{d},\boldsymbol{\epsilon}}$ (i.e. $\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}\mathcal{T}v\geq 0$).

$$\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}\mathcal{T}v(\mathbf{x})=\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{d}}\begin{cases}v(\mathbf{y}+\mathbf{a})+c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ v(\mathbf{y})+c_r & \text{otherwise}\end{cases}$$

The four possible cases are described in the following table

	$\mathbf{y}+\mathbf{a}\in\mathcal{X}$	$\mathbf{y}+\mathbf{a}\notin\mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d}\in\mathcal{X}$	Case 1	Case 3
$\mathbf{y}+\mathbf{a}+\mathbf{d}\notin\mathcal{X}$	Case 2	Case 4

- Case 1 $=0$
- Case 2 $=\Delta_{\epsilon}[v(\mathbf{y}+\mathbf{d})+c_r-v(\mathbf{y}+\mathbf{a})-c_a]$

$=\Delta_{\epsilon}\Delta_{\mathbf{d}-\mathbf{a}}v(\mathbf{y}+\mathbf{a})+\epsilon_{c_r}-\epsilon_{c_a}$

—Positive if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon}\wedge|\epsilon_{c_r}-\epsilon_{c_a}\geq 0|$

—Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})$

- Case 3 $=\Delta_{\epsilon}[v(\mathbf{y}+\mathbf{d}+\mathbf{a})+c_a-v(\mathbf{y})-c_r]$

$=\Delta_{\epsilon}\Delta_{\mathbf{d}+\mathbf{a}}v(\mathbf{y})-\epsilon_{c_r}+\epsilon_{c_a}$

—Positive if $\mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon}\wedge|\epsilon_{c_a}-\epsilon_{c_r}\geq 0|$

—Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$

- Case 4 $=0$

So $\mathcal{T}$ propagates $\mathtt{S}_{\mathbf{d},\epsilon}$ if

$$(\mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon}\wedge|\epsilon_{c_r}-\epsilon_{c_a}\geq 0|\vee\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))\wedge(\mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon}\wedge|\epsilon_{c_a}-\epsilon_{c_r}\geq 0|\vee\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}))$$

B.5. $\mathtt{PM}(\mathcal{T})$ and $\mathtt{NM}(\mathcal{T})$ (Cells 33 and 35)

$$\mathcal{T}v(\mathbf{x})-v(\mathbf{x})=\begin{cases}\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x})+c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X} \quad \text{Case 1}\\ \Delta_{\mathbf{b}}v(\mathbf{x})+c_r & \text{otherwise} \quad \text{Case 2}\end{cases}$$

So v is $\mathtt{PM}(\mathcal{T})$ if $|\Delta_{\mathbf{a}+\mathbf{b}}v\geq -c_a|\wedge(|\Delta_{\mathbf{b}}v\geq -c_r|+\mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$ and v is $\mathtt{NM}(\mathcal{T})$ if $[|\Delta_{\mathbf{a}+\mathbf{b}}v\leq -c_a|\wedge(|\Delta_{\mathbf{b}}v\leq -c_r|+\mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$

B.6. $\mathtt{IM}_{\epsilon}(\mathcal{T})$ (Cell 37)

$$\Delta_{\epsilon}\Omega_{\mathcal{T}}v(\mathbf{x})=\begin{cases}\Delta_{\epsilon}\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x})+\epsilon_{c_a} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\epsilon}\Delta_{\mathbf{b}}v(\mathbf{x})+\epsilon_{c_r} & \text{otherwise}\end{cases}$$

So, v is $\mathtt{IM}_{\epsilon}(\mathcal{T})$ if $\mathtt{S}_{\epsilon,\mathbf{a}+\mathbf{b}}\wedge|\epsilon_{c_a}\geq 0|\wedge(\mathtt{S}_{\epsilon,\mathbf{b}}\wedge|\epsilon_{c_r}\geq 0|\vee\mathtt{R}(\mathbf{a}+\mathbf{b}))$

B.7. $\mathtt{IM}_{\mathbf{d}}(\mathcal{T})$ (Cell 39)

$$\Delta_{\mathbf{d}}\Omega_{\mathcal{T}}v(\mathbf{x})=\Delta_{\mathbf{d}}\begin{cases}\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x})+c_a & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\mathbf{b}}v(\mathbf{x})+c_r & \text{otherwise}\end{cases}$$

The four possible cases are described in the following table

	$\mathbf{y}+\mathbf{a}\in\mathcal{X}$	$\mathbf{y}+\mathbf{a}\notin\mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d}\in\mathcal{X}$	Case 1	Case 3
$\mathbf{y}+\mathbf{a}+\mathbf{d}\notin\mathcal{X}$	Case 2	Case 4

- Case 1 $=\Delta_{\mathbf{d}}\Delta_{\mathbf{a}+\mathbf{b}}v(\mathbf{x})$

—Positive if $\mathtt{S}_{\mathbf{d},\mathbf{a}+\mathbf{b}}$

- Case 2 $=\Delta_{\mathbf{b}}v(\mathbf{x}+\mathbf{d})+c_r-\Delta_{\mathbf{b+a}}v(\mathbf{x})-c_a= \begin{cases}\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x})-\Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{b})+c_r-c_a\\ \Delta_{\mathbf{d-a}}\Delta_{\mathbf{b}}v(\mathbf{x}+\mathbf{a})-\Delta_{\mathbf{a}}v(\mathbf{x})+c_r-c_a\end{cases}$

  —Positive if $|\Delta_{\mathbf{a}}v\le c_r-c_a|\wedge(\mathtt{S}_{\mathbf{b},\mathbf{d}}\vee\mathtt{S}_{\mathbf{b},\mathbf{d-a}})$

  —Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a+b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))$
- Case 3 $=\Delta_{\mathbf{b+a}}v(\mathbf{x}+\mathbf{d})+c_a-\Delta_{\mathbf{b}}v(\mathbf{x})-c_r= \begin{cases}\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x})+\Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{b}+\mathbf{d})-c_r+c_a\\ \Delta_{\mathbf{d+a}}\Delta_{\mathbf{b}}v(\mathbf{x})+\Delta_{\mathbf{a}}v(\mathbf{x}+\mathbf{d})-c_r+c_a\end{cases}$

  —Positive if $|\Delta_{\mathbf{a}}v\ge c_r-c_a|\wedge(\mathtt{S}_{\mathbf{b},\mathbf{d}}\vee\mathtt{S}_{\mathbf{b},\mathbf{d+a}})$

  —Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a+b+d}}(\mathbf{a}+\mathbf{b}))$
- Case 4 $=\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x})$

  —Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$

  —Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d})\vee\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b})$

So, v is $\mathtt{IM}_{\mathbf{d}}(\mathcal{T})$ if

$$\begin{gathered}\mathtt{S}_{\mathbf{d},\mathbf{a+b}}\wedge(\mathtt{S}_{\mathbf{d},\mathbf{b}}\vee\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d})\vee\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}))\\ \wedge(|\Delta_{\mathbf{a}}v\le c_r-c_a|\wedge[\mathtt{S}_{\mathbf{d},\mathbf{b}}\vee\mathtt{S}_{\mathbf{b},\mathbf{d-a}}]\vee\mathtt{R}_{\mathbf{d},\mathbf{a+b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))\\ \wedge(|\Delta_{\mathbf{a}}v\ge c_r-c_a|\wedge[\mathtt{S}_{\mathbf{d},\mathbf{b}}\vee\mathtt{S}_{\mathbf{b},\mathbf{d+a}}]\vee\mathtt{R}_{\mathbf{d},\mathbf{a+b+d}}(\mathbf{a}+\mathbf{b}))\end{gathered}$$

Appendix C: Choice operator

$$\mathcal{C}v(\mathbf{x})=\begin{cases}\min\{v(\mathbf{y})+c_b,v(\mathbf{y}+\mathbf{a})+c_a\} & \text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ v(\mathbf{y})+c_r, & \text{otherwise}\end{cases} \tag{7}$$

with $\mathbf{y}=\mathbf{x}+\mathbf{b}$ and $\forall\mathbf{x}$, $\mathbf{x}+\mathbf{b}\in\mathcal{X}$. In this section we may use $c_d=c_a-c_b$.

C.1. Propagation of P and N (Cells 22 and 24)

We suppose that v poisitve (resp. negative). From equation (7) the condition to have $\mathcal{C}v$ positive (resp. negative) is

$$|c_a\ge 0|\wedge|c_b\ge 0|\wedge(|c_r\ge 0|\vee\mathtt{R}_{-\mathbf{b}}(\mathbf{a}))\qquad(\text{resp. } |c_a\le 0|\wedge|c_b\le 0|\wedge(|c_r\le 0|\vee\mathtt{R}_{-\mathbf{b}}(\mathbf{a})))$$

C.2. Propagation of $\mathtt{I}_\epsilon$ (Cell 26)

$$\Delta_{\epsilon}\mathcal{C}v(\mathbf{x})=\begin{cases}\Delta_{\epsilon}\min\{v(\mathbf{y})+c_b,v(\mathbf{y}+\mathbf{a})+c_a\}\\ \qquad\text{if } \mathbf{y}+\mathbf{a}\in\mathcal{X}\\ \Delta_{\epsilon}v(\mathbf{y})+c_r,\ \text{otherwise}\end{cases}$$

The four cases of $\Delta_{\epsilon}\min\{v(\mathbf{y})+c_b,v(\mathbf{y}+\mathbf{a})+c_a\}$ are described in the following table.

	$\Delta_{\mathbf{a}}v(\mathbf{y})\le -c_d$	$\Delta_{\mathbf{a}}v(\mathbf{y})\ge -c_d$
$\Delta_{\mathbf{a}}v'(\mathbf{y})\le -c_d'$	Case 1	Case 3
$\Delta_{\mathbf{a}}v'(\mathbf{y})\ge -c_d'$	Case 2	Case 4

- Case 1 $=\Delta_{\epsilon}v(\mathbf{y}+\mathbf{a})+\epsilon_{c_a}$

  —Positive if $|\epsilon_{c_a}\ge 0|$
- Case 2 $=v'(\mathbf{y})+c_b'-v(\mathbf{y}+\mathbf{a})-c_a\ge\Delta_{\epsilon}v(\mathbf{y})+\epsilon_{c_b}$

  —Positive if $|\epsilon_{c_b}\ge 0|$
- Case 3 $=v'(\mathbf{y}+\mathbf{a})+c_a'-v(\mathbf{y})-c_b\ge\Delta_{\epsilon}v(\mathbf{y}+\mathbf{a})+\epsilon_{c_a}$

  —Positive if $|\epsilon_{c_a}\ge 0|$

- Case 4 $Q = \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) + \epsilon_{c_b}$

  —Positive if $|\epsilon_{c_b} \geq 0|$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}$ (resp. $\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}$) the cases 2, 3, 4 (resp. 1, 2, 3) are Useless.

So $\mathcal{C}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if

$$\begin{pmatrix} |\epsilon_{c_a} \geq 0| \wedge |\epsilon_{c_b} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}| \wedge |\epsilon_{c_a} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}| \wedge |\epsilon_{c_b} \geq 0| \end{pmatrix} \wedge \begin{pmatrix} \mathtt{R}_{-\mathbf{b}}(\mathbf{a}) \\ \vee |\epsilon_{c_r} \geq 0| \end{pmatrix}$$

C.3. Propagation of $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (Cells 28 and 30)

We make the assumption that $\Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} v$ is positive (resp. negative) then we want find conditions on v, and $\boldsymbol{\epsilon}$ to have $\Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} \mathcal{C}$ positive (resp. negative).

$$\Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} \mathcal{C} v(\mathbf{x}) = \begin{cases} \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} \\ \qquad \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} \\ \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}), \text{ otherwise} \end{cases}$$

We focus on $\Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\}$. We use $v''(\mathbf{x})$ (resp. $c_b{}''$, $c_a{}''$) to denote $v(\mathbf{x} + 2\boldsymbol{\epsilon})$ (resp. $c_b + 2\epsilon_{c_b}$, $c_a + 2\epsilon_{c_a}$).

$$\begin{aligned} \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} = \; & \min\{v''(\mathbf{y}) + c_b{}'', v''(\mathbf{y}+\mathbf{a}) + c_a{}''\} \\ & -2\min\{v'(\mathbf{y}) + c_b{}', v'(\mathbf{y}+\mathbf{a}) + c_a{}'\} \\ & + \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} \end{aligned}$$

The 8 possible cases are given in the following table.

	$\Delta_{\mathbf{a}} v''(\mathbf{y}) \leq -c_d{}''$	$\Delta_{\mathbf{a}} v''(\mathbf{y}) \geq -c_d{}''$
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d{}'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 1	Case 5
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d{}'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 2	Case 6
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d{}'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 3	Case 7
$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d{}'$ $\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 4	Case 8

- Cases 1 and 8 are positive or negative without condition.
- Case 2 $= v''(\mathbf{y}+\mathbf{a}) + c_a{}'' - 2(v'(\mathbf{y}+\mathbf{a}) + c_a{}') + v(\mathbf{y}) + c_b = \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}} v(\mathbf{y}) - c_d$

  —Negative without condition

  —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_d} \geq 0|$
- Case 3 $= v''(\mathbf{y}+\mathbf{a}) + c_a{}'' - 2(v'(\mathbf{y}) + c_b{}') + v(\mathbf{y}+\mathbf{a}) + c_a = \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}+\mathbf{a}) + 2\Delta_{\mathbf{a}} v'(\mathbf{y}) + c_d + 2\epsilon_{c_d}$

  —Positive without condition

  —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_d} \geq 0| \vee \mathtt{S}^{ub}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_d} \leq 0|$
- Case 4 $= v''(\mathbf{y}+\mathbf{a}) + c_a{}'' - 2(v'(\mathbf{y}) + c_b{}') + v(\mathbf{y}) + c_b = \Delta_{\mathbf{a}} v''(\mathbf{y}) + \Delta_{\boldsymbol{\epsilon}} \Delta_{\boldsymbol{\epsilon}} v(\mathbf{y}) + c_d + 2\epsilon_{c_d}$

  —Negative without condition

  —Useless if $\mathtt{S}_{\mathbf{a},\boldsymbol{\epsilon}} \wedge |\epsilon_{c_d} \geq 0|$

- Case 5 $= v''(\mathbf{y}) + c_b'' - 2(v'(\mathbf{y}+\mathbf{a}) + c_a') + v(\mathbf{y}+\mathbf{a}) + c_a = \Delta_\epsilon \Delta_\epsilon v(\mathbf{y}+\mathbf{a}) - \Delta_\mathbf{a} v''(\mathbf{y}) - c_d - 2\epsilon_{c_d}$
  - —Negative without condition
  - —Useless if $\mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0|$
- Case 6 $= v''(\mathbf{y}) + c_b'' - 2(v'(\mathbf{y}+\mathbf{a}) + c_a') + v(\mathbf{y}) + c_b = \Delta_\epsilon \Delta_\epsilon v(\mathbf{y}) - 2\Delta_\mathbf{a} v'(\mathbf{y}+\mathbf{a}) - c_d - 2\epsilon_{c_d}$
  - —Positive without condition
  - —Useless if $\mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \geq 0| \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0|$
- Case 7 $= v''(\mathbf{y}) + c_b'' - 2(v'(\mathbf{y}) + c_b') + v(\mathbf{y}+\mathbf{a}) + c_a = \Delta_\epsilon \Delta_\epsilon v(\mathbf{y}) + \Delta_\mathbf{a} v(\mathbf{y}) + c_d$
  - —Negative without condition
  - —Useless if $\mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0|$

So $\mathcal{C}$ propagates $\mathtt{S}_{\epsilon,\epsilon}$ if

$$\mathtt{S}_{\mathbf{a},\epsilon} \wedge \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} = 0| \vee |\Delta_\mathbf{a} v \leq -c_d - \epsilon^+_{c_d}| \vee |\Delta_\mathbf{a} v \geq -c_d + \epsilon^-_{c_d}|$$

and propagate $\mathtt{S}^{ub}_{\epsilon,\epsilon}$ if

$$\mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \geq 0| \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0| \vee |\Delta_\mathbf{a} v \leq -c_d - \epsilon^+_{c_d}| \vee |\Delta_\mathbf{a} v \geq -c_d + \epsilon^-_{c_d}|$$

C.4. Propagation of $\mathtt{S}_{\mathbf{d},\epsilon}$ (Cell 32)

We make the assumption that v is $\mathtt{S}_{\mathbf{d},\epsilon}$ then we want find conditions on v, and $\boldsymbol{\epsilon}$ to have $\mathcal{C}$ which propagates $\mathtt{S}_{\mathbf{d},\epsilon}$.

$$\Delta_\mathbf{d} \Delta_\epsilon \mathcal{C} v(\mathbf{x}) = \Delta_\mathbf{d} \begin{cases} \Delta_\epsilon \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} \\ \Delta_\epsilon v(\mathbf{y}) + c_r, & \text{otherwise} \end{cases}$$

The 4 possible cases are given in the following table.

	$\mathbf{y}+\mathbf{a} \in \mathcal{X}$	$\mathbf{y}+\mathbf{a} \notin \mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d} \in \mathcal{X}$	Case A	Case C
$\mathbf{y}+\mathbf{a}+\mathbf{d} \notin \mathcal{X}$	Case B	Case D

C.4.1. Case A.

$$\begin{aligned} \text{Case A} &= \Delta_\epsilon \Delta_\mathbf{d} \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\} \\ &= \min\{v'(\mathbf{y}+\mathbf{d}) + c_b', v'(\mathbf{y}+\mathbf{a}+\mathbf{d}) + c_a'\} - \min\{v'(\mathbf{y}) + c_b', v'(\mathbf{y}+\mathbf{a}) + c_a'\} \\ &\quad - \min\{v(\mathbf{y}+\mathbf{d}) + c_b(, v(\mathbf{y}+\mathbf{a}+\mathbf{d}) + c_a\} + \min\{v(\mathbf{y}) + c_b', v(\mathbf{y}+\mathbf{a}) + c_a\} \end{aligned}$$

The 16 possible cases of case A are described in Table 4

- Case 1 $= \Delta_\epsilon \Delta_\mathbf{d} v(\mathbf{y}+\mathbf{a}) \geq 0$
- Case 2 $= \Delta_\mathbf{d} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) - c_d = \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) - c_d - \Delta_\mathbf{a} v'(\mathbf{y})$
  - —Positive if $|\epsilon_{c_d} \geq 0| \wedge \mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon}$
  - —Useless if $\mathtt{S}_{\mathbf{a},\mathbf{d}} \vee \mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \geq 0|$
- Case 3 $= -\Delta_{\mathbf{d}-\mathbf{a}} v(\mathbf{y}+\mathbf{a}) + \Delta_\mathbf{d} v'(\mathbf{y}+\mathbf{a}) + c_d = -\Delta_\mathbf{d} v(\mathbf{y}+\mathbf{a}) + \Delta_\mathbf{a} v(\mathbf{y}+\mathbf{d}) + \Delta_\mathbf{d} v'(\mathbf{y}+\mathbf{a}) + c_d \geq 0$
- Case 4 $= \Delta_\mathbf{d} v'(\mathbf{y}+\mathbf{a}) - \Delta_\mathbf{d} v(\mathbf{y}) \geq \begin{cases} \Delta_\mathbf{d} v(\mathbf{y}+\mathbf{a}) - \Delta_\mathbf{d} v(\mathbf{y}) \\ \Delta_\mathbf{d} v'(\mathbf{y}+\mathbf{a}) - \Delta_\mathbf{d} v(\mathbf{y}) + \underbrace{\Delta_\mathbf{a} v'(\mathbf{y}) - \Delta_\mathbf{a} v(\mathbf{y}+\mathbf{d})}_{\geq 0 \text{ if } \epsilon_{c_d} \geq 0} \\ \quad = \Delta_{\mathbf{d}+\mathbf{a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}} v(\mathbf{y}) \end{cases}$

Table 4 Possible cases for Case A $= \Delta_{\epsilon}\Delta_{\mathbf{d}} \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\}$.

Case A	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d}) \le -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y}) \le -c_d{}'$	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d}) \le -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y}) \ge -c_d{}'$	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d}) \ge -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y}) \le -c_d{}'$	$\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d}) \ge -c_d{}'$, $\Delta_{\mathbf{a}}v'(\mathbf{y}) \ge -c_d{}'$
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d}) \le -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y}) \le -c_d$	Case 1	Case 5	Case 9	Case 13
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d}) \le -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y}) \ge -c_d$	Case 2	Case 6	Case 10	Case 14
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d}) \ge -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y}) \le -c_d$	Case 3	Case 7	Case 11	Case 15
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d}) \ge -c_d$ $\Delta_{\mathbf{a}}v(\mathbf{y}) \ge -c_d$	Case 4	Case 8	Case 12	Case 16

—Positive if $\mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \ge 0|$

—Useless if $\mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \ge 0|$

- Case 5 $= \Delta_{\mathbf{d}+\mathbf{a}}v'(\mathbf{y}) - \Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a}) + c_d{}' = \Delta_{\mathbf{d}}v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a}) + c_d{}' + \Delta_{\mathbf{a}}v'(\mathbf{y}) \ge 0$
- Case 6 $= \Delta_{\mathbf{d}+\mathbf{a}}v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}}v(\mathbf{y}) + c_d{}' - c_d$

—Positive if $\mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \ge 0|$

—Useless if $\mathtt{S}_{\mathbf{d},\mathbf{a}}$

- Case 7 $= \Delta_{\mathbf{d}+\mathbf{a}}v'(\mathbf{y}) - \Delta_{\mathbf{d}-\mathbf{a}}v(\mathbf{y}+\mathbf{a}) + c_d + c_d{}'$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \le 0| \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \ge 0|$

- Case 8 $= -\Delta_{\mathbf{d}}v(\mathbf{y}) + \Delta_{\mathbf{d}+\mathbf{a}}v'(\mathbf{y}) + c_d{}' = -\Delta_{\mathbf{d}+\mathbf{a}}v(\mathbf{y}) + \Delta_{\mathbf{d}+\mathbf{a}}v'(\mathbf{y}) + \Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d}) + c_d{}'$

—Positive if $|\epsilon_{c_d} \ge 0| \wedge \mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon}$

—Useless if $\mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \ge 0|$

- Case 9 $= \Delta_{\mathbf{d}-\mathbf{a}}v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a}) - c_d{}' = \Delta_{\mathbf{d}-\mathbf{a}}v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}-\mathbf{a}}v(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{a}}v(\mathbf{y}) - c_d{}'$

—Positive if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \le 0|$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \le 0|$

- Case 10 $= \Delta_{\mathbf{d}-\mathbf{a}}v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}+\mathbf{a}}v(\mathbf{y}) + c_d{}' + c_d$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \le 0| \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \ge 0|$

- Case 11 $= -\Delta_{\mathbf{d}-\mathbf{a}}v(\mathbf{y}+\mathbf{a}) + \Delta_{\mathbf{d}-\mathbf{a}}v'(\mathbf{y}+\mathbf{a}) + c_d - c_d{}'$

—Positive if $\mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \le 0|$

—Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$

- Case 12 $= -\Delta_{\mathbf{d}}v(\mathbf{y}) + \Delta_{\mathbf{d}-\mathbf{a}}v'(\mathbf{y}+\mathbf{a}) - c_d{}'$

$= -\Delta_{\mathbf{d}}v(\mathbf{y}) - \Delta_{\mathbf{a}}v'(\mathbf{y}) + \Delta_{\mathbf{d}}v'(\mathbf{y}) - c_d \ge 0$

- Case 13 $= \Delta_{\mathbf{d}}v'(\mathbf{y}) - \Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a}) \ge \begin{cases} \Delta_{\mathbf{d}}v(\mathbf{y}) - \Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a}) \\ \Delta_{\mathbf{d}}v'(\mathbf{y}) - \Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a}) + \underbrace{\Delta_{\mathbf{a}}v'(\mathbf{y}+\mathbf{d}) - \Delta_{\mathbf{a}}v(\mathbf{y})}_{\ge 0 \text{ if } \epsilon_{c_d} \le 0} \\ \quad = \Delta_{\mathbf{d}+\mathbf{a}}v'(\mathbf{y}) - \Delta_{\mathbf{d}+\mathbf{a}}v(\mathbf{y}) \end{cases}$

—Positive if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \le 0|$

—Useless if $\mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \le 0|$

- Case 14 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d+a}} v(\mathbf{y}) + c_d = \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) - c_d \geq 0$
- Case 15 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) + c_d = \Delta_{\mathbf{d-a}} v'(\mathbf{y}) - \Delta_{\mathbf{d-a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}) + c_d$
  - —Positive if $|\epsilon_{c_d} \leq 0| \wedge \mathtt{S}_{\mathbf{d-a},\epsilon}$
  - —Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0|$
- Case 16 $= -\Delta_{\mathbf{d}} v(\mathbf{y}) + \Delta_{\mathbf{d}} v'(\mathbf{y}) \geq 0$

Note that if $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}$ or $\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}$ there is no condition because only cases 1 and 16 can be reach.

So Case A is positive if

$$
\begin{array}{cr}
|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}| \vee |\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}| \vee & \\
(|\epsilon_{c_d} \geq 0| \wedge \mathtt{S}_{\mathbf{d+a},\epsilon} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \geq 0|) & \text{(Case 2)} \\
\wedge(\mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d+a},\epsilon} \wedge |\epsilon_{c_d} \geq 0| \vee \mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \geq 0|) & \text{(Case 4)} \\
\wedge(\mathtt{S}_{\mathbf{d+a},\epsilon} \wedge |\epsilon_{c_d} \geq 0| \vee \mathtt{S}_{\mathbf{d},\mathbf{a}}) & \text{(Case 6)} \\
\wedge(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0| \vee \mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \geq 0|) & \text{(Case 7)} \\
\wedge(|\epsilon_{c_d} \geq 0| \wedge \mathtt{S}_{\mathbf{d+a},\epsilon} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \geq 0|) & \text{(Case 8)} \\
\wedge(|\epsilon_{c_d} \leq 0| \wedge \mathtt{S}_{\mathbf{d-a},\epsilon} \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0|) & \text{(Case 9)} \\
\wedge(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0| \vee \mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \geq 0|) & \text{(Case 10)} \\
\wedge(\mathtt{S}_{\mathbf{d-a},\epsilon} \wedge |\epsilon_{c_d} \leq 0| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}) & \text{(Case 11)} \\
\wedge(\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}_{\mathbf{d-a},\epsilon} \wedge |\epsilon_{c_d} \leq 0| \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0|) & \text{(Case 13)} \\
\wedge(|\epsilon_{c_d} \leq 0| \wedge \mathtt{S}_{\mathbf{d-a},\epsilon} \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \vee \mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0|) & \text{(Case 15)}
\end{array}
$$

With simplifications this condition reduces to

$$
\begin{array}{c}
|\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}| \vee |\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}| \\
\vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d-a},\epsilon} \wedge |\epsilon_{c_d} \leq 0| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d+a},\epsilon} \wedge |\epsilon_{c_d} \geq 0| \\
\vee \mathtt{S}_{\mathbf{d+a},\epsilon} \wedge \mathtt{S}_{\mathbf{d-a},\epsilon} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\epsilon} \vee \mathtt{S}_{\mathbf{a},\epsilon}) \wedge |\epsilon_{c_d} = 0|
\end{array}
$$

C.4.2. Case B.

$$
\begin{aligned}
\text{Case B} &= \Delta_\epsilon[\mathcal{C}v(\mathbf{x}+\mathbf{d}) - \mathcal{C}v(\mathbf{x})] \\
&= v'(\mathbf{y}+\mathbf{d}) - v(\mathbf{y}+\mathbf{d}) + \epsilon_{c_r} - \min\{v'(\mathbf{y}) + c_b', v'(\mathbf{y}+\mathbf{a}) + c_a'\} + \min\{v(\mathbf{y}) + c_b, v(\mathbf{y}+\mathbf{a}) + c_a\}
\end{aligned}
$$

Case B	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \leq -c_d'$,	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \geq -c_d'$
$\Delta_{\mathbf{a}} v(\mathbf{y}) \leq -c_d$	Case 1	Case 3
$\Delta_{\mathbf{a}} v(\mathbf{y}) \geq -c_d$	Case 2	Case 4

- Case 1 $= \Delta_\epsilon \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) - \epsilon_{c_a} + \epsilon_{c_r}$
  - —Positive if $\mathtt{S}_{\mathbf{d-a},\epsilon} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0|$
- Case 2 $= \Delta_{\mathbf{d-a}} v'(\mathbf{y}+\mathbf{a}) - \Delta_{\mathbf{d}} v(\mathbf{y}) - c_a' + c_b + \epsilon_{c_r} = \Delta_\epsilon \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v'(\mathbf{y}) - c_a' + c_b + \epsilon_{c_r}$
  - —Positive if $|\epsilon_{c_r} - \epsilon_{c_b} \geq 0|$
  - —Useless if $\mathtt{S}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \geq 0|$
- Case 3 $= \Delta_{\mathbf{d}} v'(\mathbf{y}) - c_b' - \Delta_{\mathbf{d-a}} v(\mathbf{y}+\mathbf{a}) + c_a + \epsilon_{c_r} = \Delta_\epsilon \Delta_{\mathbf{d-a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v'(\mathbf{y}) + c_a - c_b' + \epsilon_{c_r}$
  - —Positive if $\mathtt{S}_{\mathbf{d-a},\epsilon} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0|$
  - —Useless if $\mathtt{S}^{ub}_{\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \leq 0|$
- Case 4 $= \Delta_\epsilon \Delta_{\mathbf{d}} v(\mathbf{x}) - \epsilon_{c_b} + \epsilon_{c_r}$
  - —Positive if $|\epsilon_{c_r} - \epsilon_{c_b} \geq 0|$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}$ (resp. $\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}$) the cases 2, 3, 4 (resp. 1, 2, 3) are Useless. So case B is

- Positive if

$$\begin{array}{c} \mathtt{S}_{\mathbf{d-a},\epsilon} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \wedge |\epsilon_{c_r} - \epsilon_{c_b} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}| \wedge \mathtt{S}_{\epsilon,\mathbf{d-a}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}| \wedge |\epsilon_{c_r} - \epsilon_{c_b} \geq 0| \end{array}$$

- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a+b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})$

C.4.3. Case C.

$$\begin{aligned} \text{Case C} &= \Delta_\epsilon[\mathcal{C}v(\mathbf{x}+\mathbf{d}) - \mathcal{C}v(\mathbf{x})] \\ &= \Delta_\epsilon[\mathcal{C}v(\mathbf{x}+\mathbf{d}) - v(\mathbf{y})] - \epsilon_{c_r} \\ &= \min\{v'(\mathbf{y}+\mathbf{d}) + c_b{}', v'(\mathbf{y}+\mathbf{d}+\mathbf{a}) + c_a{}'\} \\ &\quad - \min\{v(\mathbf{y}+\mathbf{d}) + c_b, v(\mathbf{y}+\mathbf{d}+\mathbf{a}) + c_a\} \\ &\quad - v'(\mathbf{y}) + v(\mathbf{y}) - \epsilon_{c_r} \end{aligned}$$

Case C	$\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) \leq -c'$,	$\Delta_{\mathbf{a}} v'(\mathbf{y}+\mathbf{d}) \geq -c'$
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \leq -c$	Case 1	Case 3
$\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \geq -c$	Case 2	Case 4

- Case 1 $= \Delta_\epsilon \Delta_{\mathbf{d+a}} v(\mathbf{y}) + \epsilon_{c_a} - \epsilon_{c_r}$
  - Positive if $\mathtt{S}_{\epsilon,\mathbf{d+a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0|$
- Case 2 $= \Delta_{\mathbf{d+a}} v'(\mathbf{y}) - \Delta_{\mathbf{d}} v(\mathbf{y}) + c_a{}' - c_b - \epsilon_{c_r} = \Delta_\epsilon \Delta_{\mathbf{d+a}} v(\mathbf{y}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) + c_a{}' - c_b - \epsilon_{c_r}$
  - Positive if $\mathtt{S}_{\epsilon,\mathbf{d+a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0|$
  - Useless if $\mathtt{S}_{\epsilon,\mathbf{a}} \wedge \epsilon_{c_d} \geq 0|$
- Case 3 $\Delta_{\mathbf{d}} v'(\mathbf{y}) - \Delta_{\mathbf{d+a}} v(\mathbf{y}) - c_a + c_b{}' - \epsilon_{c_r} = \Delta_\epsilon \Delta_{\mathbf{d}} v(\mathbf{y}) - \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) - c_a + c_b{}' - \epsilon_{c_r}$
  - Positive if $|\epsilon_{c_b} - \epsilon_{c_r} \geq 0|$
  - Useless if $\mathtt{S}^{ub}_{\epsilon,\mathbf{a}} \wedge \epsilon_{c_d} \leq 0|$
- Case 4 $= \Delta_\epsilon \Delta_{\mathbf{d}} v(\mathbf{y}) + \epsilon_{c_b} - \epsilon_{c_r}$
  - Positive if $|\epsilon_{c_b} - \epsilon_{c_r} \geq 0|$

Note that when $\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}$ (resp.$\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}$) the cases 2, 3, and 4 (resp. 1, 2, and 3) are Useless. So case C is

- Positive if

$$\begin{array}{c} \mathtt{S}_{\epsilon,\mathbf{d+a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \wedge |\epsilon_{c_b} - \epsilon_{c_r} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}| \wedge \mathtt{S}_{\epsilon,\mathbf{d+a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}| \wedge |\epsilon_{c_b} - \epsilon_{c_r} \geq 0| \end{array}$$

- Useless if $\mathtt{R}_{\mathbf{d},\mathbf{a+b+d}}(\mathbf{a}+\mathbf{b})$

C.4.4. Case D.

$$\text{Case D} = \Delta_\epsilon[\mathcal{C}v(\mathbf{x}+\mathbf{d}) - \mathcal{C}v(\mathbf{x})] = \Delta_\epsilon \Delta_{\mathbf{d}} v(\mathbf{x}) \geq 0$$

C.4.5. Conclusion. The operator $\mathcal{C}$ propagates $\mathtt{S}_{\mathbf{d},\epsilon}$ if,

$$\begin{pmatrix} |\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}| \vee |\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}| \\ \vee \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d-a},\epsilon} \wedge |\epsilon_{c_d} \leq 0| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d+a},\epsilon} \wedge |\epsilon_{c_d} \geq 0| \\ \vee \mathtt{S}_{\mathbf{d+a},\epsilon} \wedge \mathtt{S}_{\mathbf{d-a},\epsilon} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\epsilon} \vee \mathtt{S}_{\mathbf{a},\epsilon}) \wedge |\epsilon_{c_d} = 0| \end{pmatrix}$$
$$\wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d-a},\epsilon} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \wedge |\epsilon_{c_r} - \epsilon_{c_b} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}| \wedge \mathtt{S}_{\epsilon,\mathbf{d-a}} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}| \wedge |\epsilon_{c_r} - \epsilon_{c_b} \geq 0| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a+b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\epsilon,\mathbf{d+a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \wedge |\epsilon_{c_b} - \epsilon_{c_r} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \leq -c_d - \epsilon^+_{c_d}| \wedge \mathtt{S}_{\epsilon,\mathbf{d+a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \geq 0| \\ \vee |\Delta_{\mathbf{a}} v \geq -c_d + \epsilon^-_{c_d}| \wedge |\epsilon_{c_b} - \epsilon_{c_r} \geq 0| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a+b+d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

We can simplify this results because if $|\Delta_{\mathbf{a}} \le -c_d|$ the state $\mathbf{x}+\mathbf{a}+\mathbf{b}$ is always chosen in the minimization, so the operator is equivalent to $\mathcal{T}$ (plus the cost c_a), and if $|\Delta_{\mathbf{a}} \le -c_d|$ the state $\mathbf{x}+\mathbf{a}+\mathbf{b}$ is never chosen in the minimization, so the operator is equivalent to $\mathcal{T}$ or $\mathcal{C}$ with $\mathbf{a}=\mathbf{0}$. So we can consider that $|\Delta_{\mathbf{a}} \le -c_d| = |\Delta_{\mathbf{a}} \ge -c_d| = false$. Then the relation reduces to

$$\begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \le 0| \vee \mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon} \wedge |\epsilon_{c_d} \ge 0| \\ \vee \mathtt{S}_{\mathbf{d}+\mathbf{a},\epsilon} \wedge \mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon} \wedge (\mathtt{S}^{ub}_{\mathbf{a},\epsilon} \vee \mathtt{S}_{\mathbf{a},\epsilon}) \wedge |\epsilon_{c_d} = 0| \end{pmatrix}$$
$$\wedge \begin{pmatrix} \mathtt{S}_{\mathbf{d}-\mathbf{a},\epsilon} \wedge |\epsilon_{c_r} - \epsilon_{c_a} \ge 0| \wedge |\epsilon_{c_r} - \epsilon_{c_b} \ge 0| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\epsilon,\mathbf{d}+\mathbf{a}} \wedge |\epsilon_{c_a} - \epsilon_{c_r} \ge 0| \wedge |\epsilon_{c_b} - \epsilon_{c_r} \ge 0| \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

C.5. $\mathtt{PM}(\mathcal{T})$ and $\mathtt{NM}(\mathcal{T})$ (Cells 34 and 36)

$$\mathcal{C}v(\mathbf{x}) - v(\mathbf{x}) = \begin{cases} \min\{\Delta_{\mathbf{b}} v(\mathbf{x}) + c_b, \Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + c_a\} & \text{if } \mathbf{x}+\mathbf{a}+\mathbf{b} \in \mathcal{X} \\ \Delta_{\mathbf{b}} v(\mathbf{x}) + c_r, & \text{otherwise} \end{cases}$$

So v is $\mathtt{PM}(\mathcal{C})$ if

$$(|\Delta_{\mathbf{b}} v \ge -c_b| \vee |\Delta_{\mathbf{a}} v \le -c_d|) \wedge (|\Delta_{\mathbf{a}+\mathbf{b}} v \ge -c_a| \vee |\Delta_{\mathbf{a}} v \ge -c_d|) \wedge (|\Delta_{\mathbf{b}} v \ge -c_r| \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$$

and v is $\mathtt{NM}(\mathcal{C})$ if

$$\left(|\Delta_{\mathbf{b}} v \le -c_b| \wedge \overline{|\Delta_{\mathbf{a}} v \le -c_d|} \vee |\Delta_{\mathbf{a}+\mathbf{b}} v \le -c_a| \wedge \overline{|\Delta_{\mathbf{a}} v \ge -c_d|} \right) \wedge (|\Delta_{\mathbf{b}} v \le -c_r| \vee \mathtt{R}_{-\mathbf{b}}(\mathbf{a}))$$

C.6. $\mathtt{IM}_\epsilon(\mathcal{T})$ (Cell 38)

$$\Delta_\epsilon \Omega_{\mathcal{C}} v(\mathbf{x}) = \begin{cases} \Delta_\epsilon \min\{\Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + c_a, \Delta_{\mathbf{b}} v(\mathbf{x}) + c_b\} & \text{if } \mathbf{y}+\mathbf{a} \in \mathcal{X} \\ \Delta_\epsilon \Delta_{\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_r} & \text{otherwise} \end{cases}$$

The 4 possible cases for $\Delta_\epsilon \min\{\Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + c_a, \Delta_{\mathbf{b}} v(\mathbf{x}) + c_b\}$ are given in the following table.

	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \le -c_d$	$\Delta_{\mathbf{a}} v'(\mathbf{y}) \ge -c_d$
$\Delta_{\mathbf{a}} v(\mathbf{y}) \le -c_d$	Case 1	Case 3
$\Delta_{\mathbf{a}} v(\mathbf{y}) \ge -c_d$	Case 2	Case 4

- Case 1 $= \Delta_\epsilon \Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_a}$
  — Positive if $\mathtt{S}_{\epsilon,\mathbf{b}+\mathbf{a}} \wedge |\epsilon_{c_a} \ge 0|$
- Case 2 $= \Delta_{\mathbf{a}+\mathbf{b}} v'(\mathbf{x}) + c_a{}' - \Delta_{\mathbf{b}} v(\mathbf{x}) + c_b$
  — Useless if $\mathtt{S}_{\epsilon,\mathbf{a}}$
- Case 3 $= \Delta_{\mathbf{b}} v'(\mathbf{x}) + c_b{}' - \Delta_{\mathbf{a}+\mathbf{b}} v(\mathbf{x}) - c_a \ge \Delta_\epsilon v(\mathbf{x}+\mathbf{b}) - \Delta_\epsilon v(\mathbf{x}) + \epsilon_{c_b}$
  — Positive if $\mathtt{S}_{\epsilon,\mathbf{b}} \wedge |\epsilon_{c_b} \ge 0|$
  — Useless if $\mathtt{S}^{ub}_{\epsilon,\mathbf{a}}$
- Case 4 $= \Delta_\epsilon \Delta_{\mathbf{b}} v(\mathbf{x}) + \epsilon_{c_b}$
  — Positive if $\mathtt{S}_{\epsilon,\mathbf{b}} \wedge |\epsilon_{c_b} \ge 0|$

Note that when $\Delta_{\mathbf{a}} v \le -c_d$ (resp.$\Delta_{\mathbf{a}} v \ge -c_d$) the cases 2, 3, and 4 (resp. 1, 2, and 3) are Useless. So $\Delta_\epsilon \Omega_{\mathcal{C}} v$ is positive if

$$\begin{pmatrix} \mathtt{S}_{\epsilon,\mathbf{b}} \wedge \mathtt{S}_{\epsilon,\mathbf{a}} \wedge |\epsilon_{c_a} \ge 0| \wedge |\epsilon_{c_b} \ge 0| \\ \vee |\Delta_{\mathbf{a}} v \le -c_d| \wedge \mathtt{S}_{\epsilon,\mathbf{b}+\mathbf{a}} \wedge |\epsilon_{c_a} \ge 0| \\ \vee |\Delta_{\mathbf{a}} v \ge -c_d| \wedge \mathtt{S}_{\epsilon,\mathbf{b}} \wedge |\epsilon_{c_b} \ge 0| \end{pmatrix} \wedge \begin{pmatrix} \mathtt{S}_{\epsilon,\mathbf{b}} \wedge |\epsilon_{c_r} \ge 0| \\ \vee \mathtt{R}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

C.7. $\mathtt{IM}_{\mathbf{d}}(\mathcal{T})$ (Cell 40)

$$\Delta_{\mathbf{d}}\Omega_{\mathcal{C}}v(\mathbf{x}) = \Delta_{\mathbf{d}}(\mathcal{C}v(\mathbf{x}) - v(\mathbf{x}))$$

The 4 possible cases are given in the following table.

	$\mathbf{y}+\mathbf{a}\in\mathcal{X}$	$\mathbf{y}+\mathbf{a}\notin\mathcal{X}$
$\mathbf{y}+\mathbf{a}+\mathbf{d}\in\mathcal{X}$	Case A	Case C
$\mathbf{y}+\mathbf{a}+\mathbf{d}\notin\mathcal{X}$	Case B	Case D

C.7.1. Case A.

$$\begin{aligned}\Delta_{\mathbf{d}}\Omega_{\mathcal{C}}v(\mathbf{x}) = &\min\{v(\mathbf{y}+\mathbf{d})+c_b, v(\mathbf{y}+\mathbf{d}+\mathbf{a})+c_a\} - v(\mathbf{x}+\mathbf{d})\\ &-\min\{v(\mathbf{y})+c_b, v(\mathbf{y}+\mathbf{a})+c_a\}+v(\mathbf{x})\end{aligned}$$

The 4 possible cases are given in the following table.

	$\Delta_{\mathbf{a}}v(\mathbf{y})\le -c_d$	$\Delta_{\mathbf{a}}v(\mathbf{y})\ge -c_d$
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\le -c_d$	Case 1	Case 3
$\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})\ge -c_d$	Case 2	Case 4

- Case 1 $=\Delta_{\mathbf{d}}v(\mathbf{y}+\mathbf{a})-\Delta_{\mathbf{d}}v(\mathbf{x})=\Delta_{\mathbf{d}}v(\mathbf{x}+\mathbf{b}+\mathbf{a})-\Delta_{\mathbf{d}}v(\mathbf{x})$
  — Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}}$
- Case 2 $=\Delta_{\mathbf{d}}v(\mathbf{y})-\Delta_{\mathbf{d}}v(\mathbf{x})-\Delta_{\mathbf{a}}v(\mathbf{y})-c_d\ge\Delta_{\mathbf{d}}v(\mathbf{x}+\mathbf{b})-\Delta_{\mathbf{d}}v(\mathbf{x})$
  — Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$
  — Useless if $\mathtt{S}^{ub}_{\mathbf{d},\mathbf{a}}$
- Case 3 $=\Delta_{\mathbf{d}}v(\mathbf{y})-\Delta_{\mathbf{d}}v(\mathbf{x})+\Delta_{\mathbf{a}}v(\mathbf{y}+\mathbf{d})+c_d\le\Delta_{\mathbf{d}}v(\mathbf{x}+\mathbf{b})-\Delta_{\mathbf{d}}v(\mathbf{x})$
  — Useless if $\mathtt{S}_{\mathbf{d},\mathbf{a}}$
- Case 4 $=\Delta_{\mathbf{d}}v(\mathbf{x}+\mathbf{b})-\Delta_{\mathbf{d}}v(\mathbf{x})$
  — Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$

Note that when $\Delta_{\mathbf{a}}v\le -c_d$ (resp.$\Delta_{\mathbf{a}}v\ge -c_d$) the cases 2, 3, and 4 (resp. 1, 2, and 3) are Useless. So Case A is

- Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}\wedge\mathtt{S}_{\mathbf{d},\mathbf{a}}\vee|\Delta_{\mathbf{a}}v\le -c_d|\wedge\mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}}\vee|\Delta_{\mathbf{a}}v\ge -c_d|\wedge\mathtt{S}_{\mathbf{d},\mathbf{b}}$

C.7.2. Case B.

Case B$= v(\mathbf{y}+\mathbf{d})+c_r-v(\mathbf{x}+\mathbf{d})-\mathcal{C}v(\mathbf{x})+v(\mathbf{x})$

- If $\Delta_{\mathbf{a}}v(\mathbf{y})\le -c_d$ then Case B $=\begin{cases}\Delta_{\mathbf{d}}\Delta_{\mathbf{b}}v(\mathbf{x})-\Delta_{\mathbf{a}}v(\mathbf{y})+c_r-c_a\ge\Delta_{\mathbf{b}}\Delta_{\mathbf{d}}v(\mathbf{x})+c_r+c_b\\ \Delta_{\mathbf{d}-\mathbf{a}}\Delta_{\mathbf{b}}v(\mathbf{x}+\mathbf{a})-\Delta_{\mathbf{a}}v(\mathbf{x})+c_r-c_a\end{cases}$
  — Positive if $\left(\mathtt{S}_{\mathbf{d},\mathbf{b}}\wedge|c_r+c_b\ge 0|\vee\mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}}\wedge|\Delta_{\mathbf{a}}v\le c_r-c_a|\right)$
  — Useless if $\Delta_{\mathbf{a}}v\ge -c_d$
- If $\Delta_{\mathbf{a}}v(\mathbf{y})\ge -c_d$ then Case B $=\Delta_{\mathbf{b}}\Delta_{\mathbf{d}}v(\mathbf{x})+c_r-c_b$
  — Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}\wedge|c_r-c_b\ge 0|$
  — Useless if $\Delta_{\mathbf{a}}v\le -c_d$

So Case B is

- Positive if

$$\mathtt{S}_{\mathbf{d},\mathbf{b}}\wedge|c_r+c_b\ge 0|\vee\mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}}\wedge|\Delta_{\mathbf{a}}v\le c_r-c_a|\vee|\Delta_{\mathbf{a}}v\ge -c_d|\wedge(\mathtt{S}_{\mathbf{d},\mathbf{b}}\wedge|c_r-c_b\ge 0|\vee|\Delta_{\mathbf{a}}v\le -c_d|)$$

- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})$

C.7.3. Case C. Case C $= \mathcal{C}v(\mathbf{x}+\mathbf{d}) - v(\mathbf{x}+\mathbf{d}) - v(\mathbf{y}) - c_r + v(\mathbf{x})$

- If $\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \le -c_d$ then Case C $= \begin{cases} \Delta_{\mathbf{d}}\Delta_{\mathbf{b}} v(\mathbf{x}) + \Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) - c_r + c_a \\ \Delta_{\mathbf{d}+\mathbf{a}}\Delta_{\mathbf{b}} v(\mathbf{x}) + \Delta_{\mathbf{a}} v(\mathbf{x}+\mathbf{d}) - c_r + c_a \end{cases}$

  — Positive if $|\Delta_{\mathbf{a}} v \ge c_r - c_a| \wedge (\mathtt{S}_{\mathbf{b},\mathbf{d}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}})$

  — Useless if $\Delta_{\mathbf{a}} v \ge -c_d$
- If $\Delta_{\mathbf{a}} v(\mathbf{y}+\mathbf{d}) \ge -c_d$ then Case C $= \Delta_{\mathbf{b}}\Delta_{\mathbf{d}} v(\mathbf{x}) - c_r + c_b$

  — Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_b - c_r \ge 0|$

  — Useless if $\Delta_{\mathbf{a}} v \le -c_d$

So Case C is

- Positive if $(|\Delta_{\mathbf{a}} v \ge c_r - c_a| \wedge (\mathtt{S}_{\mathbf{b},\mathbf{d}} \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}}) \vee |\Delta_{\mathbf{a}} v \ge -c_d|) \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_b - c_r \ge 0| \vee |\Delta_{\mathbf{a}} v \le -c_d|)$
- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$

C.7.4. Case D. Case D $= \Delta_{\mathbf{d}}\Delta_{\mathbf{b}} v(\mathbf{x})$

- Positive if $\mathtt{S}_{\mathbf{d},\mathbf{b}}$
- Useless if $\mathcal{X}$ is $\mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b})) \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}))$

C.7.5. Conclusion. $\Delta_{\mathbf{d}}\Omega_{\mathcal{C}} v \ge 0$ if,

$$\begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \\ \vee |\Delta_{\mathbf{a}} v \le -c_d| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}+\mathbf{a}} \\ \vee |\Delta_{\mathbf{a}} v \ge -c_d| \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}} \end{pmatrix} \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \vee \mathtt{R}_{\mathbf{d}}(\mathbf{a}+\mathbf{b}))$$
$$\wedge \begin{pmatrix} \begin{pmatrix} \mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r + c_b \ge 0| \\ \vee \mathtt{S}_{\mathbf{b},\mathbf{d}-\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \le c_r - c_a| \\ \vee |\Delta_{\mathbf{a}} v \ge -c_d| \end{pmatrix} \\ \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_r - c_b \ge 0| \vee |\Delta_{\mathbf{a}} v \le -c_d|) \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d}) \end{pmatrix} \wedge \begin{pmatrix} \begin{pmatrix} \mathtt{S}_{\mathbf{b},\mathbf{d}} \wedge |\Delta_{\mathbf{a}} v \ge c_r - c_a| \\ \vee \mathtt{S}_{\mathbf{b},\mathbf{d}+\mathbf{a}} \wedge |\Delta_{\mathbf{a}} v \ge c_r - c_a|) \\ \vee |\Delta_{\mathbf{a}} v \ge -c_d| \end{pmatrix} \\ \wedge (\mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge |c_b - c_r \ge 0| \vee |\Delta_{\mathbf{a}} v \le -c_d|) \\ \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b}) \end{pmatrix}$$

With $|\Delta_{\mathbf{a}} v \le -c_d| = |\Delta_{\mathbf{a}} v \ge -c_d| = false$ this expression reduces to

$$(|c_r \ge 0| \wedge |c_b = 0| \vee \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}+\mathbf{d})) \wedge \mathtt{S}_{\mathbf{d},\mathbf{b}} \wedge \mathtt{S}_{\mathbf{d},\mathbf{a}} \wedge \mathtt{R}_{\mathbf{d},\mathbf{a}+\mathbf{b}+\mathbf{d}}(\mathbf{a}+\mathbf{b})$$

Appendix D: Admission control

$$\begin{aligned} \boldsymbol{\mathcal{O}} v &= \mathcal{H} + \mu \mathcal{O}_0 v + \textstyle\sum_{i=1}^{n} \lambda_i \mathcal{O}_i v + p_0 v, \\ \mathcal{H}(\mathbf{x}) &= hx, \\ \mathcal{O}_0 v(\mathbf{x}) &= \mathcal{T} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = -\mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_a = c_r = 0, \end{cases} \\ \mathcal{O}_i v(\mathbf{x}) &= \mathcal{C} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = \mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_b = c_i, c_a = c_r = 0. \end{cases} \end{aligned}$$

The state space is $\mathcal{S}_1 = \mathbb{Z}^+$.

From Stidham (1985) we know that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$ and $\mathtt{I}_{\mathbf{e}_1}$.

D.1. Proof of Theorem 1

D.1.1. Monotonicity. We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{O}}$ that propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$. From Proposition 2 we obtain that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{I}_{\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$, and $\mathtt{I}_{\mathbf{e}_1}$.

$$\begin{aligned} &|\Delta_{\boldsymbol{\epsilon}}(hx) \ge 0| \\ &\bigwedge \left[\begin{matrix} |\mathcal{O}_0 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_\mu < 0| \wedge |\Omega_{\mathcal{O}_0} v \le 0| \\ \vee |\epsilon_\mu > 0| \wedge |\Omega_{\mathcal{O}_0} v \ge 0| \\ \vee |\epsilon_\mu = 0| \end{pmatrix} \end{matrix} \right] \bigwedge_{i=1}^{l} \left[\begin{matrix} |\mathcal{O}_i \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_{\lambda_i} < 0| \wedge |\Omega_{\mathcal{O}_i} v \le 0| \\ \vee |\epsilon_{\lambda_i} > 0| \wedge |\Omega_{\mathcal{O}_i} v \ge 0| \\ \vee |\epsilon_{\lambda_i} = 0| \end{pmatrix} \end{matrix} \right] \bigwedge \begin{pmatrix} |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{P}| \\ \vee |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{N}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}. \end{aligned} \tag{8}$$

From Table 3 we obtain the following relations.

- $|\Delta_{\boldsymbol\epsilon}(hx)\geq 0| = |\epsilon_h \geq 0|$
- $|\mathcal{O}_0$ propagates $\mathtt{I}_{\boldsymbol\epsilon}| = true$ (see cell 25).
- $|\Omega_{\mathcal{O}_0} v \leq 0| = |\Delta_{-\mathbf{e}_1} v \leq 0| = true$ (see cell 35).
- $|\Omega_{\mathcal{O}_0} v \geq 0| = |\Delta_{-\mathbf{e}_1} v \geq 0| = false$ (see cell 33).
- $|\mathcal{O}_i$ propagates $\mathtt{I}_{\boldsymbol\epsilon}| = |\epsilon_{c_i} \geq 0|$ because $\mathtt{R}(\mathbf{e}_1) = true$ (see cell 26).
- $|\Omega_{\mathcal{O}_i} v \leq 0| = |\Delta_{\mathbf{e}_1} v \leq 0| = false$ (see cell 36).
- $|\Omega_{\mathcal{O}_i} v \geq 0| = |\Delta_{\mathbf{e}_1} v \geq 0| = true$ (see cell 34).
- $|v$ is $\mathtt{P}| = true$ because costs are positive (see cells 21 and 22).
- $|v$ is $\mathtt{N}| = false$ because costs are not negative (see cells 23 and 24).

So equation (8) can be reduced to

$$|\epsilon_h \geq 0| \wedge |\epsilon_\mu \leq 0| \wedge |\epsilon_\eta \leq 0| \bigwedge_{i=1}^{l} \left(|\epsilon_{c_i} \geq 0| \wedge |\epsilon_{\lambda_i} \geq 0|\right) \tag{9}$$

Conclusion, the optimal value function is increasing in the arrival rates λ_i, the rejection costs c_i, the holding cost h and decreasing in the service rate μ and the discount rate η.

D.1.2. Convexity/Concavity. First we look for the condition on v and $\boldsymbol\epsilon$ to have $\boldsymbol{\mathcal{O}}$ that propagates $\mathtt{S}_{\boldsymbol\epsilon,\boldsymbol\epsilon}$. However $|\mathcal{O}_i$ propagates $\mathtt{S}_{\boldsymbol\epsilon,\boldsymbol\epsilon}| = false$, so $\boldsymbol{\mathcal{O}}$ does not propagate $\mathtt{S}_{\boldsymbol\epsilon,\boldsymbol\epsilon}$ (see Proposition 3 and cell 30 in Table 3).

Now we look for the condition on v and $\boldsymbol\epsilon$ to have $\boldsymbol{\mathcal{O}}$ that propagates $\mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}$ if the following condition is satisfied, knowing that v is $\mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$, and $\mathtt{I}_{\mathbf{e}_1}$.

$$\begin{gathered}|\Delta_{\boldsymbol\epsilon}\Delta_{\boldsymbol\epsilon}(hx)\leq 0|\\ \bigwedge\left[\begin{matrix}|\mathcal{O}_0 \text{ propagates } \mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}|\\ \bigwedge\begin{pmatrix}|\epsilon_\mu > 0|\wedge|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_0} v\leq 0|\\ \vee|\epsilon_\mu < 0|\wedge|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_0} v\geq 0|\\ \vee|\epsilon_\mu = 0|\end{pmatrix}\end{matrix}\right] \bigwedge_{i=1}^{l}\left[\begin{matrix}|\mathcal{O}_i \text{ propagates } \mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}|\\ \bigwedge\begin{pmatrix}|\epsilon_{\lambda_i} > 0|\wedge|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_i} v\leq 0|\\ \vee|\epsilon_{\lambda_i} < 0|\wedge|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_i} v\geq 0|\\ \vee|\epsilon_{\lambda_i} = 0|\end{pmatrix}\end{matrix}\right] \bigwedge\begin{pmatrix}|\epsilon_\eta > 0|\wedge|v \text{ is } \mathtt{I}_{\boldsymbol\epsilon}|\\ \vee|\epsilon_\eta < 0|\wedge|v \text{ is } \mathtt{I}_{-\boldsymbol\epsilon}|\\ \vee|\epsilon_\eta = 0|\end{pmatrix}.\end{gathered} \tag{10}$$

From Table 3 we obtain the following relations.

- $|\Delta_{\boldsymbol\epsilon}\Delta_{\boldsymbol\epsilon}(hx) \leq 0| = true$.
- $|\mathcal{O}_0$ propagates $\mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}| = true$ (see cell 29).
- $|\Delta_{-\boldsymbol\epsilon}\Omega_{\mathcal{O}_0} v \geq 0| = \mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}$ (see cell 37).
- $|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_0} v \geq 0| = \mathtt{S}_{-\boldsymbol\epsilon,\mathbf{e}_1}$ (see cell 37).
- $|\mathcal{O}_i$ propagates $\mathtt{S}_{\boldsymbol\epsilon,-\boldsymbol\epsilon}| = \mathtt{S}_{\mathbf{e}_1,\boldsymbol\epsilon} \wedge |\epsilon_{c_i} \leq 0| \vee \mathtt{S}^{ub}_{\mathbf{e}_1,\boldsymbol\epsilon} \wedge |\epsilon_{c_i} \geq 0|$ (see cell 30).
- $|\Delta_{-\boldsymbol\epsilon}\Omega_{\mathcal{O}_i} v \geq 0| = \mathtt{S}_{-\boldsymbol\epsilon,\mathbf{e}_1} \wedge |\epsilon_{c_i} \leq 0|$ (see cell 38).
- $|\Delta_{\boldsymbol\epsilon}\Omega_{\mathcal{O}_i} v \geq 0| = \mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1} \wedge |\epsilon_{c_i} \geq 0|$ (see cell 38).
- $|v$ is $\mathtt{I}_{\boldsymbol\epsilon}|$ if (see equation 9) $|\epsilon_h \geq 0| \wedge |\epsilon_\mu \leq 0| \wedge |\epsilon_\eta \leq 0| \bigwedge_{i=1}^{l} |\epsilon_{c_i} \geq 0| \wedge |\epsilon_{\lambda_i} \geq 0|$.
- $|v$ is $\mathtt{I}_{-\boldsymbol\epsilon}|$ if (see equation 9) $|\epsilon_h \leq 0| \wedge |\epsilon_\mu \geq 0| \wedge |\epsilon_\eta \geq 0| \bigwedge_{i=1}^{l} |\epsilon_{c_i} \leq 0| \wedge |\epsilon_{\lambda_i} \leq 0|$.

So equation (10) reduces to

$$\begin{pmatrix}|\epsilon_\mu > 0|\wedge\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}\\ \vee|\epsilon_\mu < 0|\wedge\mathtt{S}_{-\boldsymbol\epsilon,\mathbf{e}_1}\\ \vee|\epsilon_\mu = 0|\end{pmatrix} \bigwedge_{i=1}^{l}\left[\begin{matrix}\mathtt{S}_{\mathbf{e}_1,\boldsymbol\epsilon}\wedge|\epsilon_{c_i}\leq 0|\vee\mathtt{S}^{ub}_{\mathbf{e}_1,\boldsymbol\epsilon}\wedge|\epsilon_{c_i}\geq 0|\\ \bigwedge\begin{pmatrix}|\epsilon_{\lambda_i} > 0|\wedge\mathtt{S}_{-\boldsymbol\epsilon,\mathbf{e}_1}\wedge|\epsilon_{c_i}\leq 0|\\ \vee|\epsilon_{\lambda_i} < 0|\wedge\mathtt{S}_{\boldsymbol\epsilon,\mathbf{e}_1}\wedge|\epsilon_{c_i}\geq 0|\\ \vee|\epsilon_{\lambda_i} = 0|\end{pmatrix}\end{matrix}\right] \bigwedge|\epsilon_\eta = 0| \;. \tag{11}$$

In the following section (see equation 13) we will see that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ if

$$|\epsilon_h \geq 0| \wedge |\epsilon_{c_i} \geq 0| \wedge |\epsilon_{\lambda_i} \geq 0| \wedge |\epsilon_\mu \leq 0| \wedge |\epsilon_\eta \leq 0|,$$

so equation (11) reduces to

$$|\epsilon_{c_i} = 0| \wedge |\epsilon_{\lambda_i} = 0| \wedge |\epsilon_\mu = 0| \wedge |\epsilon_\eta = 0| \ .$$

Conclusion, the optimal value function is concave in the holding cost h.

D.1.3. Monotonicity of the optimal policy. We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{O}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ if the following condition is satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1}$, and $\mathtt{I}_{\mathbf{e}_1}$.

$$\begin{aligned}&|\Delta_{\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(hx) \geq 0|\\ &\bigwedge \begin{bmatrix} |\mathcal{O}_0 \text{ propagates } \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_\mu < 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_0} v \leq 0| \\ \vee |\epsilon_\mu > 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_0} v \geq 0| \\ \vee |\epsilon_\mu = 0| \end{pmatrix} \end{bmatrix} \bigwedge_{i=1}^{l} \begin{bmatrix} |\mathcal{O}_i \text{ propagates } \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}| \\ \bigwedge \begin{pmatrix} |\epsilon_{\lambda_i} < 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i} v \leq 0| \\ \vee |\epsilon_{\lambda_i} > 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i} v \geq 0| \\ \vee |\epsilon_{\lambda_i} = 0| \end{pmatrix} \end{bmatrix} \bigwedge \begin{pmatrix} |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{I}_{\mathbf{e}_1}| \\ \vee |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{I}_{-\mathbf{e}_1}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}.\end{aligned} \tag{12}$$

From Table 3 we obtain the following relations.

- $|\Delta_{\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(hx) \geq 0| = |\epsilon_h \geq 0|$
- $|\mathcal{O}_0$ propagates $\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}| = true$ (see cell 31).
- $|\Delta_{-\mathbf{e}_1}\Omega_{\mathcal{O}_0} v \leq 0| = \mathtt{S}_{-\mathbf{e}_1,-\mathbf{e}_1} \wedge |\Delta_{-\mathbf{e}_1} v \leq 0| = true$ (see cell 39).
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_0} v \geq 0| = \mathtt{S}_{\mathbf{e}_1,-\mathbf{e}_1} \wedge \cdots = false$ (see cell 39).
- $|\mathcal{O}_i$ propagates $\mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}}| = \mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1} \wedge |\epsilon_{c_i} \geq 0|$ (see cell 32).
- $|\Delta_{-\mathbf{e}_1}\Omega_{\mathcal{O}_i} v \geq 0| = \mathtt{S}_{-\mathbf{e}_1,\mathbf{e}_1} = false$ (see cell 40).
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_i} v \geq 0| = \mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1} = true$ (see cell 40).
- $|v$ is $\mathtt{I}_{\mathbf{e}_1}| = true$ (Stidham 1985).
- $|v$ is $\mathtt{I}_{-\mathbf{e}_1}| = false$ (Stidham 1985).

So equation (12) can be reduced, and $\boldsymbol{\mathcal{O}}$ propages $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ if

$$|\epsilon_h \geq 0| \wedge |\epsilon_{c_i} \geq 0| \wedge |\epsilon_{\lambda_i} \geq 0| \wedge |\epsilon_\mu \leq 0| \wedge |\epsilon_\eta \leq 0|. \tag{13}$$

Given that the optimal thresholds t_i decrease if

$$|\boldsymbol{\mathcal{O}} \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}| \wedge |\epsilon_{c_i} \leq 0|,$$

the optimal thresholds t_i are decreasing in the arrival rate λ_i, the holding cost h, and increasing in the service rate μ and the discount rate η.

D.2. Proof of Theorem 2

D.2.1. Effect of λ and μ : Piecewise convexity. Let $[\mu_l, \mu_u]$ (resp. $[\lambda_l, \lambda_u]$) be a set such that for all $\mu \in [\mu_l, \mu_u]$ (resp. $\lambda_i \in [\lambda_l, \lambda_u]$) the optimal thresholds S_i^* do not change. For all $\mu \in [\mu_l, \mu_u]$ (resp. $\lambda_i \in [\lambda_l, \lambda_u]$) the MDP formulation can be rewritten.

Let ϵ_μ (resp. ϵ_{λ_i}) be positive such that $\mu + \epsilon_\mu \in [\mu_l, \mu_u]$ (resp. $\lambda_i + \epsilon_{\lambda_i} \in [\lambda_l, \lambda_u]$).

- For all state space $\mathcal{X}$ and for all direction $\mathbf{a}$, $\mathcal{T}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ without conditions.
- $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{O}_0)$ is positive if v is $\mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}}$ which is true because ϵ_μ is positive. (resp. $\mathtt{IM}_{\boldsymbol{\epsilon}}(\mathcal{O}_{i>0})$ is positive if v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}$ which is true because ϵ_{λ_i} is positive.)

So $v^\star(\mathbf{x})$ is convex in $\mu \in [\mu_l, \mu_u]$ resp. $\lambda_i \in [\lambda_l, \lambda_u]$) if the optimal thresholds S_i^* do not change on the set $[\mu_l, \mu_u]$ (resp. $[\lambda_l, \lambda_u]$).

D.2.2. Effect of h and c_i : concavity and piecewise linearity. With $\epsilon_h \geq 0$ and $\epsilon_{c_i} \leq 0$, v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}}$ and operators $\mathcal{C}$ (with $\mathbf{a}=\mathbf{e}$) and $\mathcal{T}$ (with $\mathbf{a}=-\mathbf{e}$) propagate $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. So v is concave in ϵ_h and ϵ_c.

We consider a set of parameters $[h_l, h_u]$ (resp. $[c_l, c_u]$) such that the optimal thresholds S_i^* do not change on this set. As previously the MDP formulation can be rewritten on this set with translation operator only.

With $\epsilon_h \geq 0$ (resp. $\epsilon_{c_i} \geq 0$) such that $h + \epsilon_h \in [h_l, h_u]$ (resp. $c_i + \epsilon_{c_i} \in [c_l, c_u]$), then $\mathcal{T}$ propagates $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ without conditions $\forall \mathcal{X}$ and $\forall \mathbf{a}$.

Given that v $\mathtt{S}^{ub}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ imply that v is linear in $\boldsymbol{\epsilon}$, the optimal value function $v^\star(\mathbf{x})$ is linear in $h \in [h_l, h_u]$ (resp. $c_i \in [c_l, c_u]$) if the optimal thresholds S_i^* do not change on the set $[h_l, h_u]$ (resp. $[c_l, c_u]$).

Appendix E: Tandem queue, proof of Theorem 3

The optimality equations for the tandem queue problem are

$$\begin{aligned}
\boldsymbol{\mathcal{O}}v \quad &= \mathcal{H} + \mu_1 \mathcal{O}_1 v + \mu_2 \mathcal{O}_2 v + \lambda \mathcal{O}_3 v + p_0 v,\\
\mathcal{H}(\mathbf{x}) \quad &= h_1 x_1 + h_2 \max\{x_2, 0\} + b \max\{-x_2, 0\},\\
\mathcal{O}_1 v(\mathbf{x}) &= \mathcal{C} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = \mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_a = c_r = 0, \end{cases}\\
\mathcal{O}_2 v(\mathbf{x}) &= \mathcal{C} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = \mathbf{e}_2 - \mathbf{e}_1, \mathbf{b} = \mathbf{0}, \\ c_a = c_b = c_r = 0, \end{cases}\\
\mathcal{O}_3 v(\mathbf{x}) &= \mathcal{T} v(\mathbf{x}) \text{ with } \begin{cases} \mathbf{a} = -\mathbf{e}_2, \mathbf{b} = \mathbf{0}, \\ c_a = c_b = c_r = 0. \end{cases}
\end{aligned}$$

From Veatch and Wein (1992) we know that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

E.1. Monotonicity

We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{O}}$ that propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$. From Proposition 2 we obtain that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{I}_{\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

$$\begin{aligned}
&|\Delta_{\boldsymbol{\epsilon}}(h_1 x_1 + h_2 x_2^+ + b(-x_2)^+) \geq 0|,\\
&\bigwedge \begin{bmatrix} |\mathcal{O}_1 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \wedge \begin{pmatrix} |\epsilon_{\mu_1} < 0| \wedge |\Omega_{\mathcal{O}_1} v \leq 0| \\ \vee |\epsilon_{\mu_1} > 0| \wedge |\Omega_{\mathcal{O}_1} v \geq 0| \\ \vee |\epsilon_{\mu_1} = 0| \end{pmatrix} \end{bmatrix} \bigwedge \begin{bmatrix} |\mathcal{O}_2 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \wedge \begin{pmatrix} |\epsilon_{\mu_2} < 0| \wedge |\Omega_{\mathcal{O}_2} v \leq 0| \\ \vee |\epsilon_{\mu_2} > 0| \wedge |\Omega_{\mathcal{O}_2} v \geq 0| \\ \vee |\epsilon_{\mu_2} = 0| \end{pmatrix} \end{bmatrix}\\
&\bigwedge \begin{bmatrix} |\mathcal{O}_3 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \wedge \begin{pmatrix} |\epsilon_{\lambda} < 0| \wedge |\Omega_{\mathcal{O}_3} v \leq 0| \\ \vee |\epsilon_{\lambda} > 0| \wedge |\Omega_{\mathcal{O}_3} v \geq 0| \\ \vee |\epsilon_{\lambda} = 0| \end{pmatrix} \end{bmatrix} \bigwedge \begin{pmatrix} |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{P}| \\ \vee |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{N}| \\ \vee |\epsilon_\eta = 0| \end{pmatrix}.
\end{aligned} \tag{14}$$

From Table 3 we obtain the following relations.

- $|\Delta_{\boldsymbol{\epsilon}}(h_1 x_1 + h_2 x_2^+ + b(-x_2)^+) \geq 0| = |\epsilon_{h_1} \geq 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0|$,
- $|\mathcal{O}_1 \text{ propagates } \mathtt{I}_{\boldsymbol{\epsilon}}| = \mathit{true}$ (see cell 26).
- $|\Omega_{\mathcal{O}_1} v \leq 0| = \mathit{true}$ (see cell 36).

- $|\Omega_{\mathcal{O}_1} v \geq 0| = |\Delta_{\mathbf{e}_1} v \geq 0| = false$ (see cell 34).
- $|\mathcal{O}_2$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 26).
- $|\Omega_{\mathcal{O}_2} v \leq 0| = true$ (see cell 36).
- $|\Omega_{\mathcal{O}_2} v \geq 0| = |\Delta_{\mathbf{e}_2-\mathbf{e}_1} v \geq 0|$ false when $h_1 \leq h_2$ (see cell 34).
- $|\mathcal{O}_3$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}| = true$ (see cell 25).
- $|\Omega_{\mathcal{O}_3} v \leq 0| = |\Delta_{-\mathbf{e}_2} v \leq 0| = false$ (see cell 35).
- $|\Omega_{\mathcal{O}_3} v \geq 0| = |\Delta_{-\mathbf{e}_2} v \geq 0| = false$ (see cell 33).
- $|v$ is $\mathtt{P}| = true$ because all costs are positive.
- $|v$ is $\mathtt{N}| = false$ because all costs are positive.

So equation (14) can be reduced, and $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{I}_{\boldsymbol{\epsilon}}$ if

$$|\epsilon_{h_1} \geq 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0| \wedge |\epsilon_{\mu_1} \leq 0| \wedge |\epsilon_{\mu_2} \leq 0| \wedge |\epsilon_\lambda = 0| \wedge |\epsilon_\eta \leq 0|. \tag{15}$$

Conclusion, the optimal value function is increasing in the costs h_i and b, and decreasing in the service rate μ_i and the discount rate η.

E.2. Convexity/concavity

First we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{O}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$. However $|\mathcal{O}_1$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}| = false$, so $\boldsymbol{\mathcal{O}}$ does not propagate $\mathtt{S}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}}$ (see Proposition 3 and cell 30 in Table 3).

Now we look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{O}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ if the following condition is satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

$$\begin{array}{l}
|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \leq 0| \\
\bigwedge \left[\begin{array}{c} |\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge \left(\begin{array}{c} |\epsilon_{\mu_1} > 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1} v \leq 0| \\ \vee |\epsilon_{\mu_1} < 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1} v \geq 0| \\ \vee |\epsilon_{\mu_1} = 0| \end{array}\right)\end{array}\right] \bigwedge \left[\begin{array}{c} |\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge \left(\begin{array}{c} |\epsilon_{\mu_2} > 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2} v \leq 0| \\ \vee |\epsilon_{\mu_2} < 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2} v \geq 0| \\ \vee |\epsilon_{\mu_2} = 0| \end{array}\right)\end{array}\right] \\
\bigwedge \left[\begin{array}{c} |\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| \\ \bigwedge \left(\begin{array}{c} |\epsilon_{\lambda} > 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3} v \leq 0| \\ \vee |\epsilon_{\lambda} < 0| \wedge |\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3} v \geq 0| \\ \vee |\epsilon_{\lambda} = 0| \end{array}\right)\end{array}\right] \bigwedge \left(\begin{array}{c} |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{I}_{\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{I}_{-\boldsymbol{\epsilon}}| \\ \vee |\epsilon_\eta = 0| \end{array}\right).
\end{array} \tag{16}$$

From Table 3 we obtain the following relations.

- $|\Delta_{\boldsymbol{\epsilon}}\Delta_{\boldsymbol{\epsilon}}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \leq 0| = true$,
- $|\mathcal{O}_1$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| = \mathtt{S}_{\mathbf{e}_1,\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{e}_1,\boldsymbol{\epsilon}}$ (see cell 30),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1} v \leq 0| = \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_1}$ (see cell 38),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_1} v \geq 0| = \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ (see cell 38),
- $|\mathcal{O}_2$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| = \mathtt{S}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{\mathbf{e}_2-\mathbf{e}_1,\boldsymbol{\epsilon}}$ (see cell 30),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2} v \leq 0| = \mathtt{S}_{-\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$ (see cell 38),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_2} v \geq 0| = \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$ (see cell 38),
- $|\mathcal{O}_3$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}| = \mathtt{S}_{-\mathbf{e}_2,\boldsymbol{\epsilon}} \vee \mathtt{S}^{ub}_{-\mathbf{e}_2,\boldsymbol{\epsilon}}$ (see cell 29),
- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3} v \leq 0| = \mathtt{S}_{-\boldsymbol{\epsilon},-\mathbf{e}_2}$ (see cell 37),

- $|\Delta_{\boldsymbol{\epsilon}}\Omega_{\mathcal{O}_3} v \geq 0| = \mathtt{S}_{\boldsymbol{\epsilon},-\mathbf{e}_2}$ (see cell 37),
- $|v$ is $\mathtt{I}_{\boldsymbol{\epsilon}}|$ (see equation 15). $|\epsilon_{h_1} \geq 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0| \wedge |\epsilon_{\mu_1} \leq 0| \wedge |\epsilon_{\mu_2} < 0| \wedge |\epsilon_\lambda = 0| \wedge |\epsilon_\eta \leq 0|$.
- $|v$ is $\mathtt{I}_{-\boldsymbol{\epsilon}}|$ if (see equation 15) $|\epsilon_{h_1} \leq 0| \wedge |\epsilon_{h_2} \leq 0| \wedge |\epsilon_b \leq 0| \wedge |\epsilon_{\mu_1} \geq 0| \wedge |\epsilon_{\mu_2} \geq 0| \wedge |\epsilon_\lambda = 0| \wedge |\epsilon_\eta \geq 0|$.

In the following section (see equation 13) we will see that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$, $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$, and $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2}$ if

$$|\epsilon_{h_1} = 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0| \wedge |\epsilon_{\mu_1} = 0| \wedge |\epsilon_{\mu_2} = 0| \wedge |\epsilon_\lambda \leq 0|.$$

So $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}$ if

$$|\epsilon_{h_1} = 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0| \wedge |\epsilon_{\mu_1} = 0| \wedge |\epsilon_{\mu_2} = 0| \wedge |\epsilon_\lambda = 0|.$$

Conclusion, the optimal value function is concave in the costs h_2 and b.

E.3. Monotonicity of the optimal policy

We look for the condition on v and $\boldsymbol{\epsilon}$ to have $\boldsymbol{\mathcal{O}}$ that propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$ and $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$. From Proposition 3 we obtain that $\boldsymbol{\mathcal{O}}$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{d}}$ if the conditions (17) and (18) are satisfied, knowing that v is $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}$, $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_2-\mathbf{e}_1}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathtt{S}_{\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2}$, and $\mathtt{S}_{\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_1}$.

$$\begin{aligned}
&|\Delta_{\mathbf{e}_1}\Delta_{\boldsymbol{\epsilon}}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \leq 0| \\
&\bigwedge \left[\begin{array}{c} |\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| \\ \bigwedge \left(\begin{array}{c} |\epsilon_{\mu_1} > 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \leq 0| \\ \vee |\epsilon_{\mu_1} < 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \geq 0| \\ \vee |\epsilon_{\mu_1} = 0| \end{array}\right) \end{array}\right] \bigwedge \left[\begin{array}{c} |\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| \\ \bigwedge \left(\begin{array}{c} |\epsilon_{\mu_2} > 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \leq 0| \\ \vee |\epsilon_{\mu_2} < 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \geq 0| \\ \vee |\epsilon_{\mu_2} = 0| \end{array}\right) \end{array}\right] \\
&\bigwedge \left[\begin{array}{c} |\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| \\ \bigwedge \left(\begin{array}{c} |\epsilon_{\lambda} > 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \leq 0| \\ \vee |\epsilon_{\lambda} < 0| \wedge |\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \geq 0| \\ \vee |\epsilon_{\lambda} = 0| \end{array}\right) \end{array}\right] \bigwedge \left(\begin{array}{c} |\epsilon_\eta > 0| \wedge |v \text{ is } \mathtt{I}_{\mathbf{e}_1}| \\ \vee |\epsilon_\eta < 0| \wedge |v \text{ is } \mathtt{I}_{-\mathbf{e}_1}| \\ \vee |\epsilon_\eta = 0| \end{array}\right).
\end{aligned} \tag{17}$$

From Table 3 we obtain the following relations.

- $|\Delta_{\boldsymbol{\epsilon}}\Delta_{\mathbf{e}_1}(h_1x_1 + h_2x_2^+ + b(-x_2)^+) \leq 0| = |\epsilon_{h_1} \geq 0| \wedge |\epsilon_{h_2} \geq 0| \wedge |\epsilon_b \geq 0|$,
- $|\mathcal{O}_1$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \leq 0| = false$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_1} v \geq 0| = true$,
- $|\mathcal{O}_2$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_2} v \geq 0| = false$,
- $|\mathcal{O}_3$ propagates $\mathtt{S}_{\boldsymbol{\epsilon},\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \leq 0| = true$,
- $|\Delta_{\mathbf{e}_1}\Omega_{\mathcal{O}_3} v \geq 0| = false$,
- $|v$ is $\mathtt{I}_{\mathbf{e}_1}| = false$,
- $|v$ is $\mathtt{I}_{-\mathbf{e}_1}| = false$.

$$
\begin{aligned}
&|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Delta_{\epsilon}(h_1x_1+h_2x_2^+ + b(-x_2)^+)\leq 0| \\
&\bigwedge\left[\begin{matrix}|\mathcal{O}_1 \text{ propagates } \mathtt{S}_{\epsilon,\mathbf{e}_2-\mathbf{e}_1}| \\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_1}>0|\wedge|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1}v\leq 0| \\ \vee|\epsilon_{\mu_1}<0|\wedge|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1}v\geq 0| \\ \vee|\epsilon_{\mu_1}=0|\end{pmatrix}\end{matrix}\right]\bigwedge\left[\begin{matrix}|\mathcal{O}_2 \text{ propagates } \mathtt{S}_{\epsilon,\mathbf{e}_2-\mathbf{e}_1}| \\ \bigwedge\begin{pmatrix}|\epsilon_{\mu_2}>0|\wedge|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2}v\leq 0| \\ \vee|\epsilon_{\mu_2}<0|\wedge|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2}v\geq 0| \\ \vee|\epsilon_{\mu_2}=0|\end{pmatrix}\end{matrix}\right] \\
&\bigwedge\left[\begin{matrix}|\mathcal{O}_3 \text{ propagates } \mathtt{S}_{\epsilon,\mathbf{e}_2-\mathbf{e}_1}| \\ \bigwedge\begin{pmatrix}|\epsilon_{\lambda}>0|\wedge|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3}v\leq 0| \\ \vee|\epsilon_{\lambda}<0|\wedge|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3}v\geq 0| \\ \vee|\epsilon_{\lambda}=0|\end{pmatrix}\end{matrix}\right]\bigwedge\begin{pmatrix}|\epsilon_{\eta}>0|\wedge|v \text{ is } \mathtt{I}_{\mathbf{e}_2-\mathbf{e}_1}| \\ \vee|\epsilon_{\eta}<0|\wedge|v \text{ is } \mathtt{I}_{\mathbf{e}_1-\mathbf{e}_2}| \\ \vee|\epsilon_{\eta}=0|\end{pmatrix}.
\end{aligned} \tag{18}
$$

From Table 3 we obtain the following relations.

- $|\Delta_{\epsilon}\Delta_{\mathbf{e}_2-\mathbf{e}_1}(h_1x_1+h_2x_2^+ + b(-x_2)^+)\leq 0| = |\epsilon_{h_1}\leq 0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|$,
- $|\mathcal{O}_1$ propagates $\mathtt{S}_{\epsilon,\mathbf{e}_2-\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1}v\leq 0| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_1}v\geq 0| = false$,
- $|\mathcal{O}_2$ propagates $\mathtt{S}_{\epsilon,\mathbf{e}_2-\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2}v\leq 0| = false$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_2}v\geq 0| = true$,
- $|\mathcal{O}_3$ propagates $\mathtt{S}_{\epsilon,\mathbf{e}_2-\mathbf{e}_1}| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3}v\leq 0| = true$,
- $|\Delta_{\mathbf{e}_2-\mathbf{e}_1}\Omega_{\mathcal{O}_3}v\geq 0| = false$,
- $|v$ is $\mathtt{I}_{\mathbf{e}_2-\mathbf{e}_1}| = false$,
- $|v$ is $\mathtt{I}_{-\mathbf{e}_2-\mathbf{e}_1}| = false$.

So equations (17) and (18) reduce to

$$|\epsilon_{h_1}=0|\wedge|\epsilon_{h_2}\geq 0|\wedge|\epsilon_b\geq 0|\wedge|\epsilon_{\mu_1}=0|\wedge|\epsilon_{\mu_2}=0|\wedge|\epsilon_{\lambda}\leq 0|.$$

Conclusion, the optimal switching curves $s_i(x_1)$ are increasing in the demand rate λ, the backlog costs b, and decreasing in the holding cost h_2.

References

Aktaran-Kalaycı, T., H. Ayhan. 2009. Sensitivity of optimal prices to system parameters in a steady-state service facility. *European Journal of Operational Research* **193**(1) 120–128.

Aydin, S., Y. Akçay, F. Karaesmen. 2009. On the structural properties of a discrete-time single product revenue management problem. *Operations Research Letters* **37**(4) 273–279.

Benjaafar, S., J.-P. Gayon, S. Tepe. 2010. Optimal control of a productioninventory system with customer impatience. *Operations Research Letters* **38**(4) 267–272.

Çil, E., E.L. Örmeci, F. Karaesmen. 2009. Effects of system parameters on the optimal policy structure in a class of queueing control problems. *Queueing Systems* **61** 273–304.

Gans, N., S. Savin. 2007. Pricing and capacity rationing for rentals with uncertain durations. *Management Science* **53**(3) 390–407.

Koole, G. 1998. Structural results for the control of queueing systems using event-based dynamic programming. *Queueing Systems* **30**(3) 323–339.

Koole, G. 2006. Monotonicity in markov reward and decision chains: Theory and applications. *Foundations and Trends in Stochastic Systems* **1** 176.

Ku, C.-Y., S. Jordan. 1997. Access control to two multiserver loss queues in series. *IEEE Transactions on Automatic Control* **42**(7) 1017–1023.

Lippman, S. A. 1975. Applying a new device in the optimization of exponential queuing systems. *Operations Research* **23**(4) 687–710.

Müller, A. 1997. How does the value function of a markov decision process depend on the transition probabilities? *Mathematics of Operations Research* **22**(4) 872–885.

Ozkan, C., F. Karaesmen, S. Ozekici. 2013. Structural properties of Markov modulated revenue management problems. *European Journal of Operational Research* **225**(2) 324–331.

Satir, B., S. Savasaneril, Y. Serin. 2012. Pooling through lateral transshipments in service parts systems. *European Journal of Operational Research* **220**(2) 370–377.

Smith, J. E., K. F. McCardle. 2002. Structural properties of stochastic dynamic programs. *Operations Research* **50**(5) 796–809.

Stidham, Shaler Jr. 1985. Optimal control of admission to a queueing system. *IEEE Transactions on Automatic Control* **30**(8) 705– 713.

Veatch, M. H., L. M. Wein. 1992. Monotone control of queueing networks. *Queueing Systems* **12**(3) 391–408.

Veatch, M. H., L. M. Wein. 1994. Optimal control of a Two-Station tandem Production/Inventory system. *Operations Research* **42**(2) 337–350.

Weber, R. R., S. Jr. Stidham. 1987. Optimal control of service rates in networks of queues. *Advances in Applied Probability* **19**(1) 202–218.

Zerhouni, H., J.-P. Gayon, Y. Frein. 2013. Control of a production-inventory system with product returns dependent on customer demand. *International Journal of Production Economics* **143**(1) 62–71.

Zhuang, W., M. Z.F. Li. 2012. Monotone optimal control for a class of markov decision processes. *European Journal of Operational Research* **217**(2) 342–350.