CHAPTER 2

Geometrical Optics

2.1 Introduction

In Chap. 1, we discussed the solution of the wave equation and developed techniques to analyze different phenomena and devices based on the wave aspects of light, such as reflection and refraction of light and optical interference coatings. We will use these techniques in later chapters to investigate other important wave effects in photonics, including diffractive spreading of confined laser beams, light propagation in optical fibers, pulse broadening in dispersive media, nonlinear coupling of waves, and so on. In fact, we can also employ wave methods in the analysis of imaging systems. For example, suppose that we place a candle in front of a converging lens and want to find the location and magnification of the image. After all, light emanating from each point of the candle can be described by electromagnetic waves and Maxwell's equations give us the full recipe to determine how light will propagate through the optical system. This approach falls within the scope of diffraction theory which we will not discuss here. In such a case, keeping track of the electromagnetic field distribution between the object and its image can turn out to be quite a difficult task and often requires numerical computation. However, if the wavelength of light is much smaller than the characteristic system dimensions, such as the length of the candle or the aperture of the lens, as in this particular example, the location and the magnification of the image can be determined by using a much simpler approach based on geometrical optics. This is what we will be exploring in this chapter.

In geometrical optics, wave aspects of light such as wavelength, electric field distribution, and polarization are not taken into account. Light is simply modeled as consisting of rays. As such, it is assumed to travel in a straight line without spreading as long as propagation takes place in a homogeneous medium. The direction of the ray is the same as the direction of the Poynting vector for the corresponding light wave. The ray direction will change only when a discontinuity such as a dielectric interface or a lens is encountered. In this case, Snell's law [see Eq. (1.209)] can be used to find the new direction of the ray. Alternatively, ray trajectories can also be computed by application of the Fermat principle which states that light will follow the specific trajectory along which the transit time is minimized. This approach will not be discussed here. Let us emphasize once again how simplified the geometrical optic approach is: we will make no reference to the wavelength, electric field strength, or the polarization of light waves, and simply keep track of the direction of the Poynting vector. The method works especially well in the calculation of image position and magnification so long as the wavelength is much smaller than the system dimensions.

In this chapter, we will develop the methods of geometrical optics and apply them to the analysis of common optical systems. Typically, the components of an optical system are centered on an optical axis. If the light ray coincides with the optical axis, it goes through the system without an offset or deflection. As an arbitrary light ray moves through the optical system, its height r and angle of inclination θ will change with respect to the optical axis. In this chapter, we will concentrate on the case where θ is very small so that the approximation $\sin\theta \approx \tan\theta \approx \theta$ is always valid. This regime is commonly referred to as paraxial geometrical optics. In this case, nonlinear terms involving the trigonometric functions of θ are negligible and the effect of the optical system can be described by linear transformations on r and θ. Specifically, each optical component can be modeled by a 2×2 matrix also referred to as the ABCD matrix. In Sec. 2.2, we will introduce the ABCD matrices and describe how the matrix elements can be determined. We will then discuss how the image location and magnification can be determined in the paraxial limit. By using the concepts of the principal planes, we will also show in Sec. 2.5 that a general paraxial optical system can be represented in terms of displacements and a thin lens. The general representation of a paraxial system developed in this chapter will be used later in Sec. 3.3 to derive the transformation rule for a Gaussian light beam. Ray tracing in continuous media, the Eikonal equation, graded-index rods, and stability criteria for optical resonators will also be discussed. Further discussion of geometrical optics can be found in Refs. 1 to 8 listed at the end of the chapter.

2.2 Matrix Formulation of Paraxial Optics

Let us consider a general optical system sketched in Fig. 2.1. The z axis corresponds to the optical axis. The input and output planes of the optical system are located at $z = z_i$ and $z = z_o$, respectively. So, as an example, we may have a telescope between $z = z_i$ and $z = z_o$ as shown in Fig. 2.1. In this particular case, the telescope consists of two lenses of focal lengths f_1 and f_2 separated by a distance d. An incident light ray enters the system at $z = z_i$ and exists at $z = z_o$. We are interested in developing a simple method to determine how the characteristics of the light ray will be modified by the optical system.

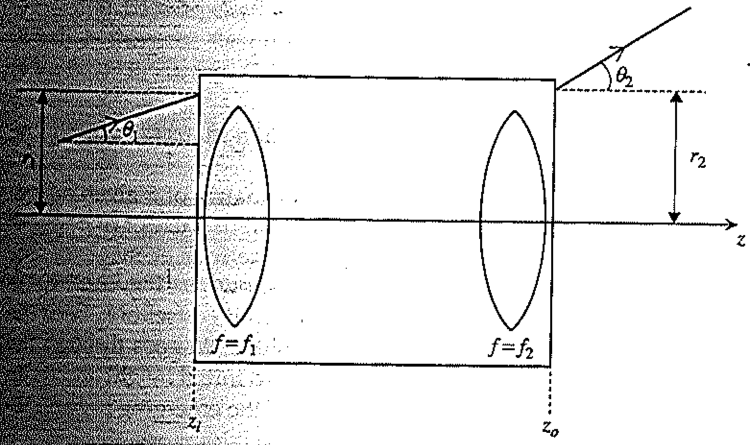


Figure 2.1 Sketch of a general optical system. The incident ray has the height r_1 and the slope angle θ_1 at the input plane $z = z_i$. The ray emerging at the output plane $z = z_o$ has the height r_2 and the slope angle θ_2.

In general, two parameters are required to completely describe a ray of light: its height (shown as r_1 on the input side of Fig. 2.1) and slope (θ_1 on the input side of Fig. 2.1) with respect to the optical axis. It is therefore convenient to describe this ray as a 2×1 column vector:

$$\bar{r} = \begin{bmatrix} r \\ \theta \end{bmatrix} \tag{2.1}$$

An optical system will in general modify these ray parameters so that the emerging ray at the output plane will have a different height r_2 and a different slope θ_2. Our goal is to determine how the optical system maps the input ray to the output ray. Hence, we will represent the optical system with a set of transformation rules which, given an initial ray height and slope, produces the final height and slope. Provided that we stay in the paraxial optics regime, where the rays maintain a small angle of inclination with the optical axis, this transformation can be represented by a 2×2 matrix $\mathbf{M}_T$ with elements that are independent of the ray height and slope. The parameters of the output ray are hence related to those of the input ray according to

$$\bar{r}_f = \mathbf{M}_T \bar{r}_i \tag{2.2}$$

where $\bar{r}_f$ and $\bar{r}_i$ are the respective final and initial ray vectors given by

$$\bar{r}_f = \begin{bmatrix} r_f \\ \theta_f \end{bmatrix} \quad \text{and} \quad \bar{r}_i = \begin{bmatrix} r_i \\ \theta_i \end{bmatrix} \tag{2.3}$$

In Eq. (2.2), M_T is a 2×2 matrix conventionally expressed as

$$M_T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.4}$$

We will refer to M_T as the ray transformation matrix or simply the ABCD matrix.

How do we determine the ray transfer matrix of a given optical system? Since M_T is a 2×2 matrix, it turns out that use of two special input rays is sufficient to fully determine the four elements of M_T. These rays, designated as $\vec{\rho}_1$ and $\vec{\rho}_2$, are shown in Fig. 2.2. Ray $\vec{\rho}_1$ (Fig. 2.2*a*) is parallel to the optical axis at a finite offset of r_0 from the optical axis, whereas the second ray $\vec{\rho}_2$ (Fig. 2.2*b*) goes through the optical axis at the input plane $z = z_i$ with zero initial height and finite slope angle θ_0. In general, the parameters of the emerging ray are different for these two cases.

Expanding the ray transformation given in Eq. (2.2), we get the following explicit expressions for r_f and θ_f:

$$r_f = Ar_i + B\theta_i \tag{2.5}$$

$$\theta_f = Cr_i + D\theta_i \tag{2.6}$$

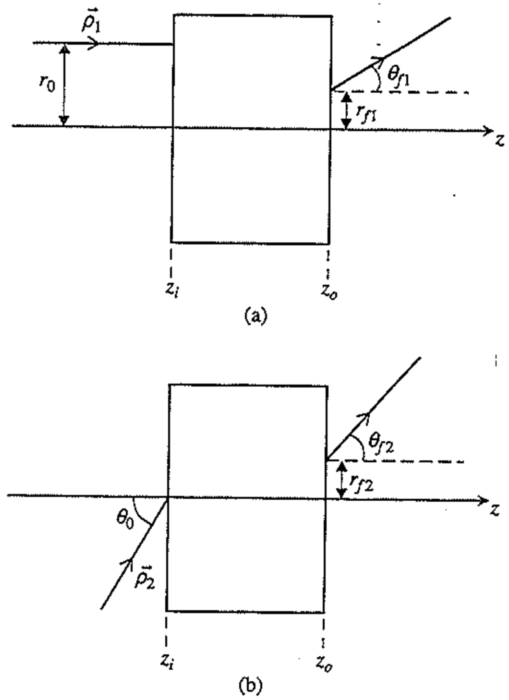


Figure 2.2 Sketch of the special incident rays (*a*) $\vec{\rho}_1$ and (*b*) $\vec{\rho}_2$ which are used to determine the elements of the ray transfer matrix M...

Let us first consider the ray $\vec{\rho}_1$ which has a finite height and zero slope ($\theta_i = 0$). In this case, the output ray has a height of r_{f1} and a slope angle of θ_{f1}. Since $\theta_i = 0$, Eqs. (2.5) and (2.6) reduce to

$$A = \frac{r_{f1}}{r_0} \quad \text{and} \quad C = \frac{\theta_{f1}}{r_0} \tag{2.7}$$

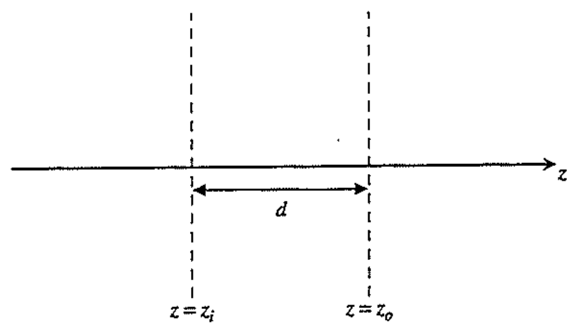
immediately, giving the elements A and C of M_T. Similarly, the initial height is zero for the second ray $\vec{\rho}_2$, in which case, Eqs. (2.5) and (2.6) can be used to determine B and D from

$$B = \frac{r_{f2}}{\theta_0} \quad \text{and} \quad D = \frac{\theta_{f2}}{\theta_0} \tag{2.8}$$

By using the special rays $\vec{\rho}_1$ and $\vec{\rho}_2$, the four elements of the ray transformation matrix M_T are determined. Once M_T is known, the effect of the optical system on any arbitrary ray can be calculated by using Eq. (2.2). Note that the choice of the reference plane locations is critical. If z_i and z_o are changed, the elements of M_T will change. Below, we provide several examples to demonstrate the application of this technique.

Example The ray transformation matrix for a displacement of length d
We start with the simplest possible example and use the above method to determine the ray transformation matrix corresponding to a displacement of length d shown in Fig. 2.3. Here, the optical system which is located between the input ($z = z_i$) and output ($z = z_o$) planes is simply a segment of space of length d.

First, consider the ray vector $\vec{\rho}_1$ which corresponds to a parallel ray incident at a height of r_0 (see Fig. 2.4*a*). Since there is only free space, the ray will emerge parallel to the optical axis at the same height, so that $r_{f1} = r_0$ and $\theta_{f1} = 0$. This gives $A = 1$ and $C = 0$ using Eq. (2.7). In the case of $\vec{\rho}_2$, where the incident ray has an arbitrary slope of θ_0 and an initial height of $r_1 = 0$ (Fig. 2.4*b*), the emerging ray will have the same slope ($\theta_{f2} = \theta_0$) and the ray height will be $r_{f2} = d\tan\theta_0 \approx d\theta_0$. Note that here, we have used the paraxial approximation by assuming that $\tan\theta_0 \approx \theta_0$.



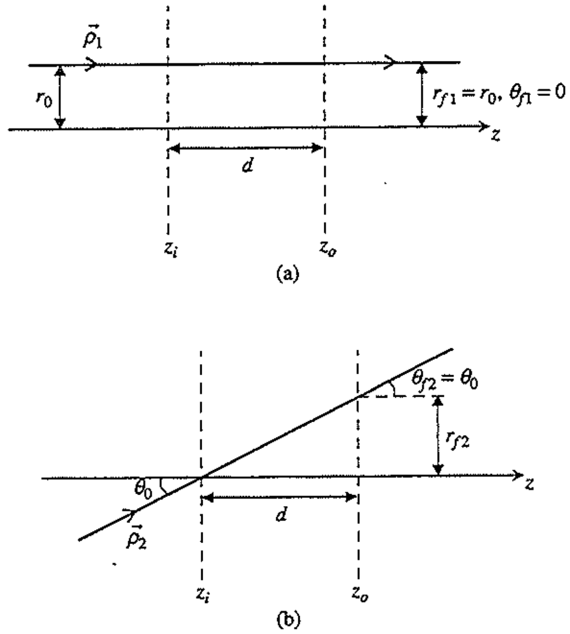


Figure 2.4 Use of the special rays (*a*) $\bar{\rho}_1$ and (*b*) $\bar{\rho}_2$ in the determination of the ray transformation matrix for a displacement of *d*.

By using Eq. (2.8), this gives B = *d* and D = 1. Therefore, the ray transfer matrix of a displacement of length *d* is given by

$$M_T = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \tag{2.9}$$

Note that the determinant of this matrix is unity. We will find that if the index of refraction of the medium on both sides of the optical system is the same, the determinant will always be unity. This is a useful fact to check whether or not you have calculated the ray transfer matrix properly.

Example The ray transformation matrix for a thin lens of focal length *f*
As a second example, let us derive the ray transformation matrix for a thin lens of focal length *f*. Under the thin-lens aproximation, we simply neglect the finite thickness of the lens. The input and output reference planes are chosen to coincide with the front and back surfaces, respectively. Figures 2.5*a* and *b* show the action of the thin lens on the rays $\bar{\rho}_1$ and $\bar{\rho}_2$.

First, consider the incident ray vector $\bar{\rho}_1$. It is well known from elementary optics that a parallel ray of light passing through a thin lens of focal length *f* is bent to cross the optical axis at the focal point (shown as *F* in Fig. 2.5*a*), which is at a distance of *f* from the lens. Since the thin lens has a negligible thickness, the ray height remains essentially the same, so $r_{f1} = r_0$. On the other hand, the ray is bent downward and the new slope becomes $\theta_{f1} \approx \tan\theta_{f1} = -r_0 / f$. This gives A = 1 and C = −1/*f* using Eq. (2.7). Similarly, the effect of the thin lens on $\bar{\rho}_2$ can be analyzed by using the fact that a light ray going through the center of a thin lens emerges without any deflection (see Fig. 2.5*b*). In this case, we have $r_{f2} = 0$ and $\theta_{f2} = \theta_0$. Hence, B = 0 and D = 1 from Eq. (2.8). Hence, the overall ray transfer matrix for the thin lens becomes

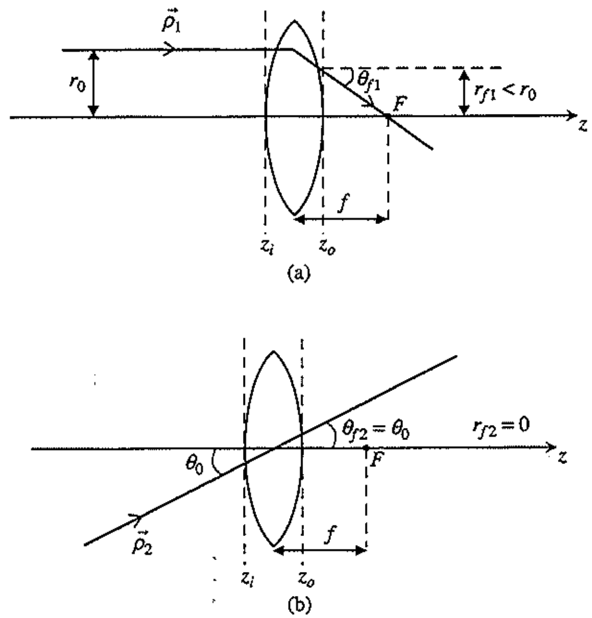


Figure 2.5 Effect of a thin lens of focal length *f* on the special rays (*a*) $\bar{\rho}_1$ and (*b*) $\bar{\rho}_2$.

$$M_T = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \tag{2.10}$$

Another important optical system is the spherical refracting surface shown in Figs. 2.6*a* and *b*. Here, the two sides of the curved interface have different indices of refraction and this leads to bending of the light rays. The procedure outlined above can also be used in this case to derive the ray transformation matrix. We leave it as an exercise for the reader (Prob. 2.1) to show that M_T has the form

$$M_T = \begin{bmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix} \tag{2.11}$$

where *R* is the radius of curvature of the interface, and n_1 and n_2 are the indices of refraction on the input and output sides. Here, it is also necessary to establish a sign convention for the radius of the surface. For a ray propagating in the positive *z*-direction, we take the radius of curvature *R* to be positive if the center of curvature is to the right of the curved surface as in Fig. 2.6*a*, and negative otherwise (Fig. 2.6*b*). When *R* is negative as in Fig. 2.6*b*, an incident parallel ray will diverge for $n_2 > n_1$. This can be easily verified by the application of Snell's law. Also note that when the refractive index before and after the optical system is not the same, the determinant of the corresponding ray transformation matrix

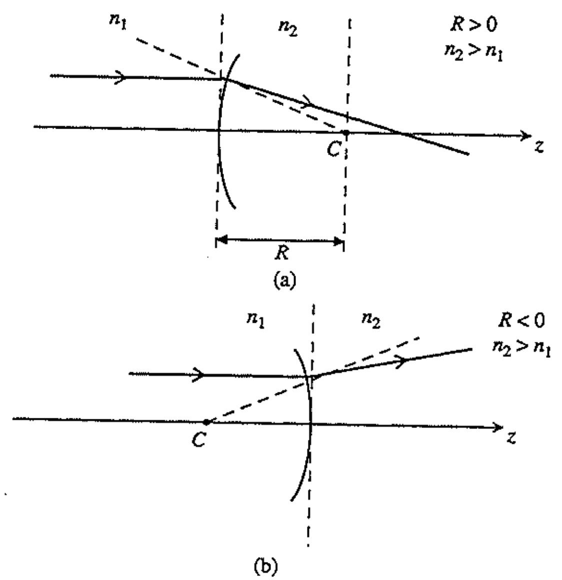


Figure 2.6 Effect of a curved refracting surface with a (*a*) positive and (*b*) negative radius on a parallel ray of light. For both cases, $n_2 > n_1$.

is no longer unity. In general, the determinant AD-BC becomes (n_1/n_2), where n_1 and n_2 are the indices of refraction on the input and output sides of the optical system, respectively. In the next section, we will use the ray transformation matrix of the curved refracting surface to obtain an expression for the focal length of a thin lens in terms of the radii of curved surfaces and the refractive index of the lens material.

2.3 Cascaded Optical Systems

A general optical system such as a telescope may consist of several curved surfaces, displacements, and so on. This is represented as a block diagram in Fig. 2.7. In such a case, the overall ray transfer

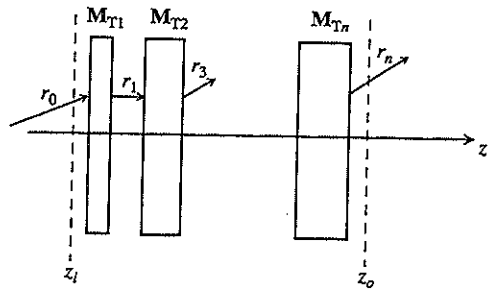


Figure 2.7 Block diagram of a cascaded optical system. The ray transformation matrix for the composite system is the product of the matrices for the individual elements. The correct ordering of matrix multiplication is shown in Eq. (2.13).

matrix $\mathbf{M}_T$ of the composite system is the product of the transformation matrices representing the individual optical elements. The order in which the matrices are multiplied is particularly important. To see this better, let us refer to Fig. 2.7 and assume that the light ray is propagating in the positive z-direction. The incident ray represented by $\bar{r}_0$ will first encounter the optical element with the ray transformation matrix $\mathbf{M}_{T1}$. The emerging ray $\bar{r}_1$ will be given by $\bar{r}_1 = \mathbf{M}_{T1}\bar{r}_0$. Now, $\bar{r}_1$ is incident on the second optical element, and the output ray $\bar{r}_2$ will become $\bar{r}_2 = \mathbf{M}_{T2}\bar{r}_1 = \mathbf{M}_{T2}\mathbf{M}_{T1}\bar{r}_0$. It is clear that the ray $\bar{r}_n$ emerging from the *n*th element will be related to the incident ray through

$$\bar{r}_n = \mathbf{M}_T\bar{r}_0 \tag{2.12}$$

where

$$\mathbf{M}_T = \mathbf{M}_{Tn}\mathbf{M}_{Tn-1}\dots\mathbf{M}_{T1} = \prod_{i=n}^{1}\mathbf{M}_{Ti} \tag{2.13}$$

Hence, this analysis shows that we can describe the entire paraxial optical system with one 2 × 2 matrix. When several matrices are involved, it is always a good strategy to check the determinant of the overall matrix. This should be unity if the refractive index before and after the optical system is the same. The next two examples illustrate the use of Eq. (2.13) in the analysis of cascaded optical systems.

Example Consider the optical system consisting of two thin lenses L_1 and L_2 and a displacement of *d* in between, as shown in Fig. 2.8. $f_1 = 10$ cm, $f_2 = 20$ cm, and $d = 8$ cm. The ray incident on L_1 has a height of 3 mm and a slope angle of 1°. Calculate the height and the slope angle of the output ray immediately after L_2.

Solution The input and output planes of the cascaded optical system are located at L_1 and L_2. First, note that we need to express the incidence angle in radians so that the approximation $\sin\theta \approx \tan\theta \approx \theta$ applies. A slope angle of 1° corresponds to 0.017 radian. Expressing lengths in cm and angles in radians, the incident ray vector becomes

$$\bar{r}_0 = \begin{bmatrix} 0.3 \\ 0.017 \end{bmatrix}$$

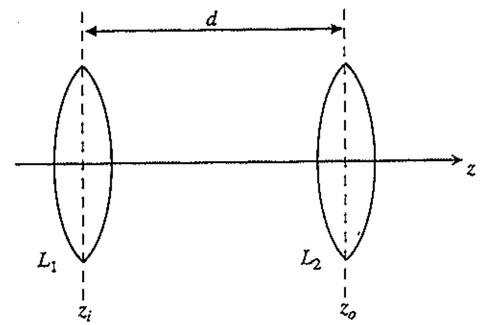


Figure 2.8 Example involving a cascaded optical system.

The cascaded optical system consists of the lens L_1, followed by the displacement d which is followed by the lens L_2. Therefore, the ray transformation matrix $\mathbf{M}_T$ which describes the optical system between z_i and z_o becomes

$$\mathbf{M}_T = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -0.05 & 1 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 8 \\ -0.11 & 0.6 \end{bmatrix}$$

Note that the determinant of $\mathbf{M}_T$ is unity as expected. The output ray $\bar{r}_f$ becomes

$$\bar{r}_f = \begin{bmatrix} 0.2 & 8 \\ -0.11 & 0.6 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.0175 \end{bmatrix} = \begin{bmatrix} 0.2 \\ -0.0225 \end{bmatrix}$$

So, the output ray has a height of 2 mm and a slope angle of −1.29°.

Example Lens maker's formula

Equation (2.13) can be used to obtain a useful expression for the focal length of a thin lens in terms of the radii of curvature of the surfaces and the refractive index of the lens material. Consider the thin lens sketched in Fig. 2.9. It consists of two curved surfaces with radii R_1 and R_2 and the refractive index of the material is n. We can treat this lens as a cascaded optical system composed of three optical elements: a curved refracting surface with a radius of R_1, followed by a displacement of d, followed by a second refracting surface of radius R_2. We will make the thin lens approximation and neglect the thickness of the lens. Using Eq. (2.13), we can then write the ray transformation matrix $\mathbf{M}_T$ as

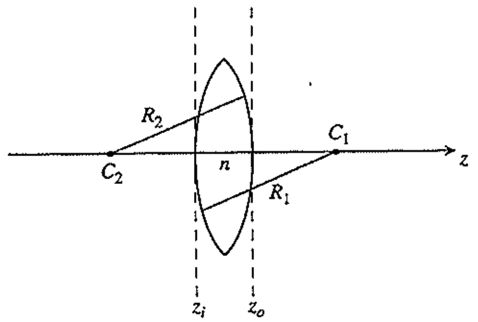
$$\mathbf{M}_T = \begin{bmatrix} 1 & 0 \\ \frac{(n-1)}{R_2} & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{(1-n)}{nR_1} & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) & 1 \end{bmatrix} \quad (2.14)$$

Above, we have used the ray transfer matrix of a curved refracting surface given in Eq. (2.11). Because the thickness of the lens is neglected, the second matrix in Eq. (2.14) becomes the identity matrix. Note that the result of Eq. (2.14) looks identical to the ray transformation matrix of a thin lens whose focal length f is given by

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad (2.15)$$

Equation (2.15) is the well-known "lens maker's formula" that is widely used in geometrical optics. This formula simply tells us that a dielectric slab bounded by spherical surfaces will act as a lens and its focusing characteristics are related to the refractive index of the medium, and the radii of the curved surfaces. Furthermore, by carefully choosing the signs of these radii, it is possible to make lenses with positive (convex) as well as negative (concave) effective focal lengths. The inverse of the focal length gives the focusing power of the lens, typically measured in m^{-1} or diopters. It is fulfilling to see that the matrix formulation of paraxial optics can be used to obtain this important result in an elegant way.

Figure 2.9 Sketch of a thin lens consisting of spherical refracting surfaces.



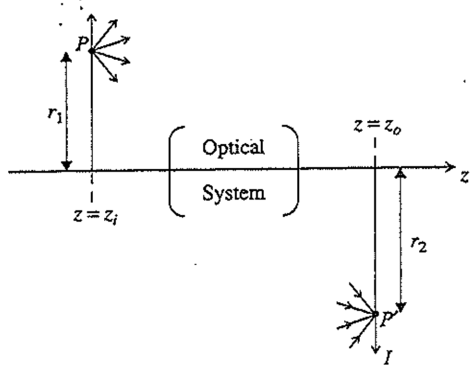
2.4 Imaging by a Paraxial Optical System

Using the matrix formalism of paraxial optics, we can determine the location of an image and the magnification produced by an optical system. In this section, we want to establish a criterion for finding the image location. Consider Fig. 2.10 where an optical system creates the image of an object as shown. Let us assume that the ray transformation matrix $\mathbf{M}_T$ connecting the object to the image is given by

$$\mathbf{M}_T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (2.16)$$

We are interested in finding a general relation satisfied by the elements of $\mathbf{M}_T$ when the object and the image are located at the input and output reference planes, respectively, as shown in Fig. 2.10. To obtain this relation, consider rays originating from a particular point P on the object. P is at a distance of r_1 from the optical axis. Every ray originating from P converges to a point P' on the image, irrespective

Figure 2.10 Imaging in a general paraxial optical system.



of the initial slope angle θ_1 at P. If P' is at a distance of r_2 from the optical axis, then, by definition, r_2 and r_1 are related by

$$r_2 = \mathrm{A}r_1 + \mathrm{B}\theta_1 \tag{2.17}$$

Since r_2 is independent of θ_1, B must be zero. This is an important result in imaging calculations. One can determine the location of the image by requiring that the element B of the ray transformation matrix between the object and the image planes vanish. In addition, notice that when the imaging condition is satisfied, the matrix element A gives the magnification of the optical system, in other words, the ratio of the image height to the object height.

Example To illustrate the versatility of the above method, let us revisit the cascaded optical system example shown in Fig. 2.8. An object with a height of 4 mm is placed at a distance of 30 cm from L_1 as shown in Fig. 2.11. Calculate the location and the height of the image.

Solution The ray transformation matrix of the cascaded system shown in Fig. 2.8 and consisting of the lenses L_1 and L_2, separated by d, was calculated earlier. Expressing all distances in centimeters, the ray transformation matrix $\mathbf{M}_T$ connecting the image and the object is given by

$$\mathbf{M}_T = \begin{bmatrix} 1 & d_I \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 8 \\ -0.11 & 0.6 \end{bmatrix} \begin{bmatrix} 1 & 30 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.2-0.11d_I & 14-2.7d_I \\ -0.11 & -2.7 \end{bmatrix}$$

At the location of the image, B = 0 and hence $d_I = 14/2.7 = 5.2$ cm. To find the height of the image, we calculate A at $d_I = 5.2$ cm. This gives A = −0.37. The negative sign of A indicates that the image is inverted. Since the initial height of the object was 4 mm, the image height becomes 1.48 mm (inverted).

Example Imaging by a thin lens
Let us use the imaging result to find the general relation between the image and object positions in the case of a thin lens. Figure 2.12 shows an object which is placed at a distance of s_1 from a thin lens of focal length f. We are interested in finding the distance s_2 of the image from the thin lens and the resulting magnification. We leave it as an exercise (Prob. 2.5) for the reader to show that the ray transformation matrix from the object to the image is given by

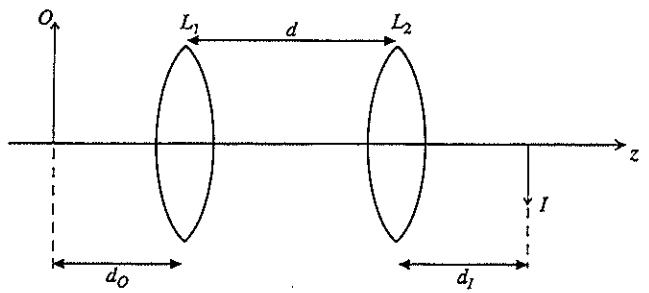


Figure 2.11 Example of imaging by a cascaded optical system.

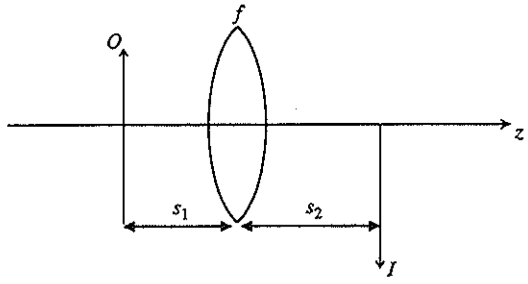


Figure 2.12 Imaging by a thin lens.

$$\mathbf{M}_T = \begin{bmatrix} 1-\frac{s_2}{f} & s_1\left(1-\frac{s_2}{f}\right)+s_2 \\ -\frac{1}{f} & 1-\frac{s_1}{f} \end{bmatrix} \tag{2.18}$$

The condition B = 0 readily gives the relation between s_1 and s_2, a result again well known in elementary optics:

$$\frac{1}{s_2} = \frac{1}{f} - \frac{1}{s_1} \tag{2.19}$$

Furthermore, the magnification becomes

$$\mathrm{A} = 1-\frac{s_2}{f} = -\frac{s_2}{s_1} \tag{2.20}$$

The above example also shows that one does not need to remember many of the specific formulas of imaging introduced in courses on elementary optics. The condition B = 0 provides a general approach for imaging calculations in any paraxial optical system. If the output side of the lens is immersed in a medium with refractive index n, the distance s_2' of the image becomes ns_2, where s_2 is given by Eq. (2.19). The effect of having a different refractive index on the output side of the imaging system is further explored in Prob. 2.6.

2.5 General Decomposition of a Paraxial Optical System

In this section, we show that an arbitrary paraxial system can always be decomposed into a cascaded optical system consisting of a thin lens sandwiched between two displacements. We will derive this important result by introducing the concept of principal planes and focal points. The parameters of the equivalent cascaded system will be expressed in terms of the elements of the ray transfer matrix. The results obtained here will be used later in the next chapter to derive the transformation rule for Gaussian beams.

2.5.1 Second Principal Plane and Second Focal Point

Let us consider a general paraxial optical system shown in Fig. 2.13. In general, an incident ray parallel to the optical axis leaves the system at an arbitrary slope angle and height which are different from the initial values. Note that we could, in principle, produce the same output ray height and slope angle by using a thin lens. How can we determine the focal length and the location of this lens? To answer this question, we assume that the incident light ray moves parallel to the optical axis up to an imaginary plane H_2. At H_2, it is bent and crosses the optical axis at the point F_2. H_2 and F_2 are called the second principal plane and the second focal point, respectively, of the optical system. H_2 simply gives the location of this effective thin lens that will produce the same output ray height and the output slope angle as the original optical system. We can easily obtain expressions for the positions of H_2 and F_2 by defining the following displacements:

δ_{F2} = displacement from the output plane $z = z_o$ to the focal point F_2
f_2 = displacement from the second principal plane H_2 to the second focal point F_2
δ_2 = displacement from the output plane to the second principal plane H_2

Because the incident light ray is parallel to the optical axis, $\theta_1 = 0$, giving $r_2 = Ar_1$ and $\theta_2 = Cr_1$. From Fig. 2.13, one therefore sees that

$$\delta_{F2} = -\frac{r_2}{\theta_2} = -\frac{A}{C} \tag{2.21}$$

$$f_2 = -\frac{r_1}{\theta_2} = -\frac{1}{C} \tag{2.22}$$

and

$$\delta_2 = \delta_{F2} - f_2 = \frac{(1-A)}{C} \tag{2.23}$$

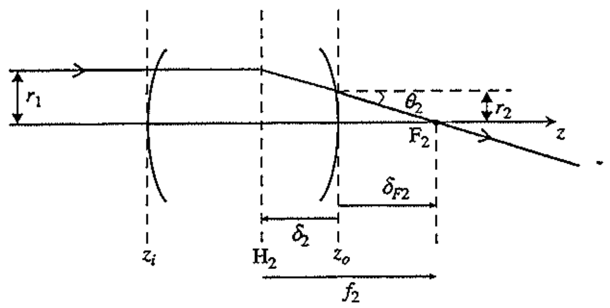


Figure 2.13 Second principal plane and second focal length of a paraxial optical system.

To summarize, a thin lens with an effective focal length of f_2, placed at H_2 has the same action on an incident parallel light ray as the original system. Note that the displacements given in Eqs. (2.21) to (2.23) can be positive or negative, depending on the specific configuration of the optical system. Hence, it is possible to have H_2 on either side of the output plane. Similarly, the effective focal length f_2 of the optical system can be positive or negative.

2.5.2 First Principal Plane and First Focal Point

We next investigate an oblique incident ray as shown in Fig. 2.14. If the ray passes through a special point, shown as F_1, it exits the system parallel to the optical axis. F_1 is defined as the first focal point. As before, we want to determine the focal length and location of a thin lens which would produce the same effect. We assume that the light ray moves in a straight line up to an imaginary plane H_1, known as the first principal plane. At H_1, the ray is refracted to move parallel to the optical axis. To find the location of H_1 and F_1, we define the following displacements:

δ_{F1} = displacement from the input plane z_i to the first principal point F_1
f_1 = displacement from the first principal focal point F_1 to the first principal plane H_1
δ_1 = displacement from the input plane z_i to the first principal plane H_1

We leave it as an exercise (Prob. 2.7) for the reader to show that δ_{F1}, f_1, and δ_1 are given by

$$\delta_{F1} = \frac{D}{C} \tag{2.24}$$

$$f_1 = -\frac{(AD-BC)}{C} \tag{2.25}$$

$$\delta_1 = \frac{D-(AD-BC)}{C} \tag{2.26}$$

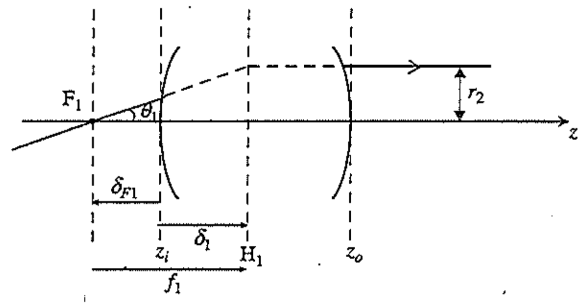


Figure 2.14 First principal plane and first focal point of a paraxial optical system.

Note that if the refractive index before and after the optical system is the same, then AD − BC = 1 and $f_1 = f_2 = -1/C$.

2.5.3 Ray Transformation Matrix Between H_1 and H_2

By using Figs. 2.13 and 2.14, we see that the ray transformation matrix $\mathbf{M}_T(H_1 \to H_2)$ between H_1 and H_2 consists of three components:

1. Displacement of $-\delta_1$ from H_1 to the input plane z_i
2. The optical system between z_i and z_o
3. Displacement of δ_2 from z_o to H_2

Written in full, $\mathbf{M}_T(H_1 \to H_2)$ becomes

$$\mathbf{M}_T(H_1 \to H_2) = \begin{pmatrix} 1 & \delta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -\delta_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{1-A}{C} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -\frac{D-(AD-BC)}{C} \\ 0 & 1 \end{pmatrix} \quad (2.27)$$

$$= \begin{pmatrix} 1 & 0 \\ C & AD-BC \end{pmatrix}$$

Recall from before that $AD - BC = n_1/n_2$, where n_1 and n_2 are the respective refractive indices at the input and output planes of the optical system. Hence, $\mathbf{M}_T(H_1 \to H_2)$ becomes

$$\mathbf{M}_T(H_1 \to H_2) = \begin{pmatrix} 1 & 0 \\ C & \frac{n_1}{n_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \quad (2.28)$$

The above equation summarizes a very important result about paraxial optical systems. In the most general case, where the input and output planes of an optical system have different refractive indices, the ray transformation matrix between the principal planes is equivalent to that of a cascaded optical system consisting of a dielectric interface followed by a thin lens of focal length $f = -1/C$. If the input and exit planes both have the same refractive index, then the ray transfer matrix reduces to that of a thin lens of focal length $f = -1/C$. In fact, one can use this as an alternative method in imaging calculations as the following example illustrates.

Example Let us solve the imaging problem shown in Fig. 2.11 by using the method of principal planes. Recall that the ray transformation matrix $\mathbf{M}_T$ between the input and output planes was calculated to be

$$\mathbf{M}_T = \begin{bmatrix} 0.2 & 8 \\ -0.11 & 0.6 \end{bmatrix}$$

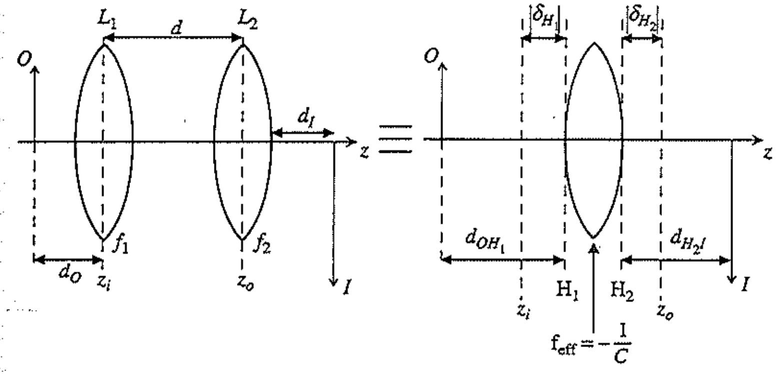


Figure 2.15 Application of principal planes to imaging problems.

By using Eqs. (2.26) and (2.23), we calculate δ_{H_1} and δ_{H_2} to determine the locations of the principal planes H_1 and H_2:

$$\delta_{H_1} = \frac{D-(AD-BC)}{C} = \frac{0.6-1}{-0.11} = 3.64 \text{ cm}$$

$$\delta_{H_2} = \frac{1-A}{C} = \frac{1-0.2}{-0.11} = -7.27 \text{ cm}$$

So, we can represent the optical system with a thin lens of effective focal length $f_{eff} = -1/C = 9.09$ cm, situated between H_1 and H_2 as shown in Fig. 2.15. The distance d_{OH_1} between the object and the thin lens becomes $d_{OH_1} = 3.64 + 30 = 33.64$ cm. The image distance d_{H_2I} can be calculated by using the imaging relation for a thin lens:

$$\frac{1}{d_{OH_1}} + \frac{1}{d_{H_2I}} = \frac{1}{f_{eff}}$$

This gives $d_{H_2I} = 12.46$ cm. So, the distance of the image from the second output plane becomes 12.46 − 7.27 = 5.2 cm, as calculated before. Finally, the magnification M is given by

$$M = -\frac{d_{H_2I}}{d_{OH_1}} = -0.37$$

2.5.4 Representation of a Paraxial Optical System in Terms of Displacements and a Lens

Let us consider a general paraxial optical system described by the ray transfer matrix

$$\mathbf{M}_T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (2.29)$$

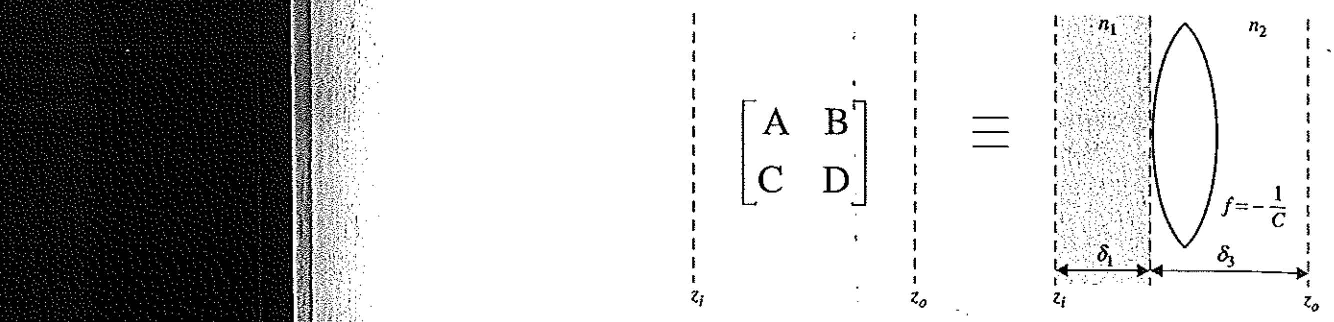


Figure 2.16 General representation of a paraxial optical system.

By using Eqs. (2.27) and (2.28), M_T can be expressed as

$$M_T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \delta_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} 1 & \delta_1 \\ 0 & 1 \end{pmatrix} \tag{2.30}$$

where

$$\delta_3 = -\delta_2 = \frac{A-1}{C} \tag{2.31}$$

and

$$\delta_1 = \frac{D-(AD-BC)}{C} \tag{2.32}$$

In other words, a general paraxial optical system can always be represented as a cascaded system consisting of a displacement (δ_1), a dielectric interface, a thin lens of focal length $f = -1/C$, and a second displacement (δ_3). This is schematically shown in Fig. 2.16. In the next chapter, we will use this result to derive the general transformation rule for the q-parameter of Gaussian beams.

2.6 Geometrical Optics in Continuous Media

In the previous sections, we analyzed the effect of discrete optical systems on the propagation of light rays. Here, we will consider optical systems where the refractive index changes continuously in space. We will assume during the analysis that, as before, the z-axis defines the optical axis and the change in the refractive index occurs in the radial direction as shown in Fig. 2.17. As can be seen from Fig. 2.17, the ray makes an angle of θ with the optical axis at the point A which is at a distance of r from the optical axis. The refractive index at A is $n(r)$. After propagating an infinitesimal distance dz along the optical axis, the ray moves to point B which is $r + dr$ away from the axis and the angle made with the optical axis becomes $\theta + d\theta$. Because n changes continuously with r, the ray will in general follow a curved

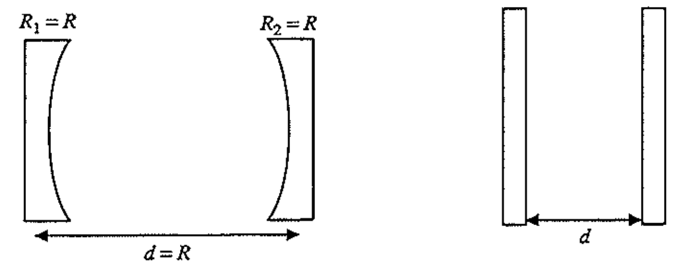


FIGURE 2.26 Examples of conditionally stable resonators: (a) a confocal cavity and (b) plane-mirror cavity.

Points on the boundaries of the stability region, i.e. where $g_1 g_2 = 0$ or 1, represent cavity configurations which are conditionally stable. Two examples are worth mentioning. Consider the origin of the graph with $g_1 = g_2 = 0$ (point O in Fig. 2.25). This point corresponds to the confocal cavity with $d = R_1 = R_2 = R$ sketched in Fig. 2.26*a*. The confocal cavity, so named because the foci of the mirrors are exactly coincident, is widely used as an optical spectrum analyzer as we will discuss later in Chap 3. Point A, on the other hand, with $g_1 = g_2 = 1$ corresponds to a cavity with two plane mirrors, since $R_1 = R_2 = \infty$ in this case. This is sketched in Fig. 2.26*b*. In both cases, the mirrors have to be exactly parallel in order to confine the ray inside the resonator.

Problems

2.1 (*a*) Derive Eq. (2.11) for the the ray transformation matrix of a curved dielectric refracting surface with a radius of curvature of R.

(*b*) Obtain the ray transformation matrix for the flat dielectric interface shown in Fig. 2.27.

2.2 What is the ray transformation matrix for a dielectric slab of refractive index n and length d?

2.3 Derive the ray transformation matrix for the optical system consisting of a thin lens of focal length f followed by a displacement of d.

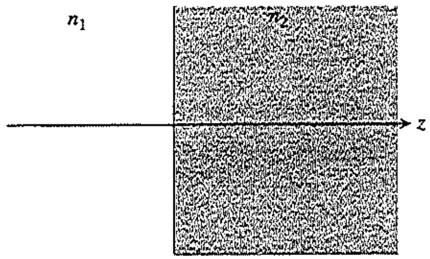


FIGURE 2.27 Flat dielectric interface for Prob. 2.1(*b*).

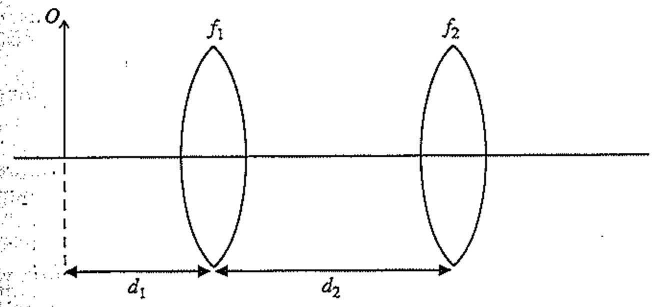


FIGURE 2.28 Optical system for Prob. 2.4.

2.4 Use matrix methods to find the image location and magnification in the optical system shown in Fig. 2.28 ($f_1 = 5$ cm, $f_2 = 8$ cm, $d_1 = 4$ cm, $d_2 = 7$ cm).

2.5 Derive the ray transformation matrix from the object to the image given in Eq. (2.18).

2.6 An object is placed at a distance of s_1 from the input plane of a paraxial optical system. Show that if the input and output sides of the imaging system have the same refractive index, then the distance s_2 of the image from the output plane is given in general by

$$s_2 = -\frac{As_1 + B}{Cs_1 + D}$$

Also show that if the input and the output sides of the imaging system have different refractive indices of n_1 and n_2, then the image distance s_2' becomes

$$s_2' = \frac{n_2}{n_1} s_2$$

2.7 Derive the expressions for the displacements δ_{F1}, f_1, and δ_1 given in Eqs. (2.24) to (2.26).

2.8 (*a*) Find the effective focal length of a ball lens of radius R and refractive index n. Here, the focal length corresponds to the distance between one of the principal planes and the corresponding focal point.

(*b*) What is the working distance of the ball lens, i.e., how far is the focal point from the exit end of the ball lens?

2.9 Newton's imaging equation

(*a*) What is the ray transfer matrix between the focal points F_1 and F_2?

(*b*) Suppose that an object is placed such that the displacement from the first focal point to the object is d_1. In addition, d_2 is the displacement from the second focal point to the image. Show that

$$d_1 d_2 = -f_1 f_2.$$

(*c*) Show that in the special case where $n_1 = n_2$ one recovers the Newton's imaging equation

$$d_1 d_2 = -f^2.$$

(*d*) Express the magnification in terms of d_1 and f_1.

2.10 Show that in the paraxial limit, a concave mirror with a radius of R is equivalent to a thin lens of focal length f given by $f = R/2$.

2.11 Derive the ray transformation matrix for the two-mirror resonator given in Eq. (2.69).

2.12 Suppose that a convex reflecting mirror with a radius of curvature of R is placed in front of a flat mirror at a distance of d. Investigate the stability of the resonator that is formed.

2.13 Show that the trace (A + D) of the ray transformation matrix which represents one round-trip of the two-mirror cavity is independent of the starting point.

2.14 Thermal lensing in a cylindrical dielectric rod: Solid-state crystals and glasses which are used in lasers are often susceptible to thermal lensing. Here, heat load due to unused pump power produces a temperature distribution inside the medium leading to a position dependent refractive index. The thermally loaded rod then behaves like a lens. In this problem, we will use geometrical optics to investigate how the strength of thermal lensing varies with power. For further discussion of thermal lensing, see Refs. 9 to 11.

(*a*) Consider a cylindrical dielectric rod of length L_0 and radius r_0. Assume that heat generation rate per unit volume is h (units = W/cm^3). Show that in a cylindrically symmetric geometry, the steady-state temperature distribution $T(r)$ satisfies the differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial r^2}=-\frac{h}{\kappa}$$

where κ is the heat conductivity of the medium.

(*b*) Assume that h is a constant. Also, assume that heat conduction in the longitudinal direction is negligible. Find the temperature distribution $T(r)$ as a function of r. Take $T(r_0) = T_b$, where T_b is the boundary temperature.

(*c*) What is the maximum temperature rise inside the crystal? Note that the maximum temperature occurs at $r = 0$.

(*d*) Typically, the refractive index n of a dielectric varies as a function of temperature T according to

$$n(T) = n_0 + n_T(T - T_r)$$

where T_r is some reference temperature at which $n = n_0$ and $n_T = \frac{dn}{dT}$ is the thermal index coefficient. In the presence of the heat source h, find the effective quadratic index coefficient β. Take $T_r = T_b$ and assume that $n(r)$ departs by a small amount from n_0.

(*e*) Show that at low pump powers, the effective thermal focal length varies inversely with the absorbed pump power.

2.15 Find the eigenvalues and the eigenvectors [Eq. (2.68)] of the round-trip ray transformation matrix M_T for the case where AD − BC = 1 and the stability condition is met. By using standard diagonalization techniques of matrix algebra, show that the mth power of M_T is given by Eq. (2.69).

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