Chapter 10: Dynamics of Rotational Motion:

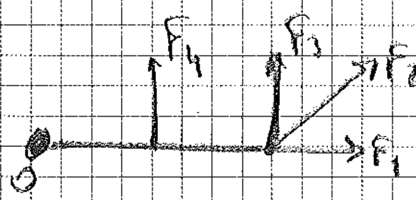
We will find the relationships that govern the dynamics of the rotational motion.

For linear motion $\sum \vec{F} = m\vec{a}$

For rotational motion ?

We first define the torque (or moment).

Torque: The tendency of a force to cause or change the rotational motion of a body.



Consider a rigid body rotating around the point O.

$\vec{F}_1$ would not cause any rotational motion.

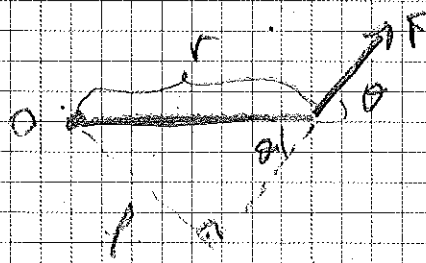
If the magnitudes of $\vec{F}_2$, $\vec{F}_3$ and $\vec{F}_4$ are equal

$\vec{F}_3$ would lead to larger angular acceleration than $\vec{F}_2$ and $\vec{F}_4$.

⇒ We define: $\boxed{\tau = F\ell}$ torque of F with respect to O.

F: magnitude of the applied force

ℓ: perpendicular distance between the point $\vec{F}$ is applied and O.



$\Rightarrow \tau = F\ell = Fr\sin\theta$

If there are more than 1 force acting on a body, total torque is given as

$$\boxed{\tau = \tau_1 + \tau_2 + \tau_3 \ldots}$$

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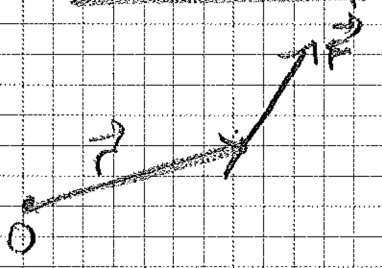
In vector notation we define the torque vector as

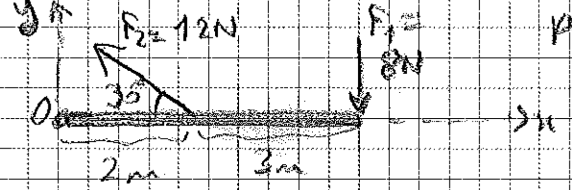
$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$ cross product operation.



$\vec{r}$: Position vector from O to the point P where force is applied

Note that torque can be negative, positive or equal to 0.

Problem 10.2: Calculate the net torque about point O.



$\vec{F}_1 = -8N\hat{j}$, $\vec{F}_2 = 12N\left(-\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$

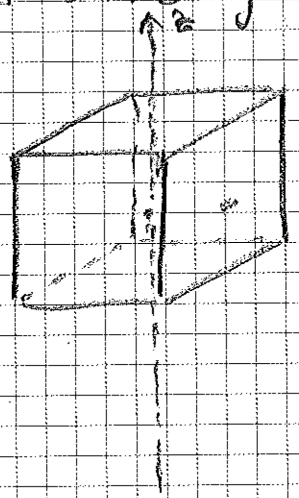
$$\Rightarrow \vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = (5m\hat{i}) \times (-8N\hat{j}) + (2m\hat{i}) \times (12N)\left(-\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$$

$$= -40(Nm)\hat{k} + 12(Nm)\hat{k}$$

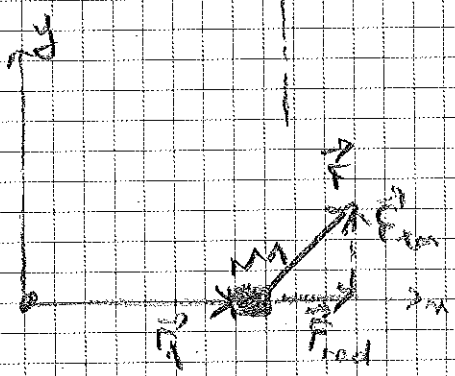
$$\boxed{\vec{\tau} = -28(Nm)\hat{k}}$$

10.2. Torque and Acceleration for a Rigid Body:

Consider a rotating rigid body.



We take the z-axis along the direction of the axis of rotation.



Consider a small mass element of the cube.
Let a force $\vec{F}$ be applied to the small mass element.

⇒ Torque with respect to the axis of rotation:

$$\vec{\tau}_1 = \underbrace{\vec{r}_1}_{\text{along the x direction}} \times \left(\vec{F}_{tan} + \vec{F}_{rad} + \vec{F}_z\right) = \vec{r}_1 \times \vec{F}_{tan}$$

Relationship between tangential acceleration and angular acceleration:

$$a_{tan} = r_1 \underbrace{\alpha_z}_{\text{angular acceleration}} \quad , \quad F_{tan} = m_1 a_{tan} = m_1 r_1 \alpha_z$$

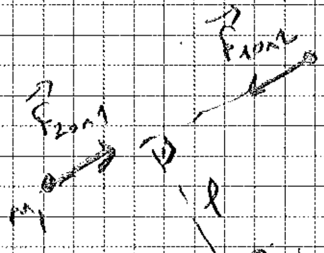
$$\tau_z = r_1 m_1 r_1 \alpha_z = m_1 r_1^2 \alpha_z$$

⇒ Total torque in the direction of the axis of rotation:

$$\sum \tau_z = \underbrace{\left(\sum m_i r_i^2\right)}_{I} \alpha_z = I \alpha_z \Rightarrow \boxed{\sum \tau_z = I \alpha_z}$$

$\boxed{\sum \tau_z = I\alpha_z}$: The component of the net torque in the direction of the axis of rotation is equal to the body's moment of inertia about the rotation axis multiplied with the angular acceleration.

An important consequence:



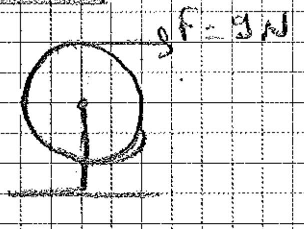
According to Newton's 3rd law
$\vec{F}_{2on1} = -\vec{F}_{1on2}$

$\Rightarrow$ The torque due to $\vec{F}_{2on1}$ with respect to O is negative of the torque of $\vec{F}_{1on2}$ with respect to O.

$\Rightarrow$ Therefore, all the internal torques add to zero.

$\Rightarrow$ $\sum \tau_z$ will only include the torques due to external forces.

Ex 10.2:



A cylinder has a diameter of 0.12m and mass $M = 50kg$.

Cable wrapped to the cylinder is unwinded with a force of 9N.

What is the acceleration of the cable?

$$I_{cyl} = \frac{1}{2} MR^2$$ R → radius

$$\sum \tau_z = I\alpha_z$$

$$\Rightarrow \sum \tau_z = (9N) \times (0.06m)$$

$$I_{cyl} = \frac{1}{2}(50kg)(0.06m)^2$$

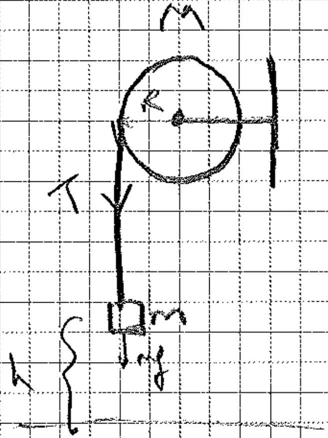
$$\Rightarrow \alpha_z = \frac{(9N) \times (0.06)}{\frac{1}{2}(50kg)(0.06)^2} = \frac{18}{(50)(0.06)} = 6\,rad/s^2$$

⇒ Acceleration of the cable:

$$a_x = R\alpha_z = 0.36\, m/s^2$$

Ex 10.3:

A mass m is attached to a cylinder via a massless rope.
What is the acceleration of m when it hits the ground?



$$\sum \tau_z = I\alpha_z$$

$$TR = \frac{1}{2}MR^2\alpha_z$$

$$\Rightarrow \frac{2T}{MR} = \alpha_z \Rightarrow a_y = \alpha_z R = \frac{2T}{M}$$

$$\Rightarrow T = \frac{Ma_y}{2}$$

$$\Rightarrow mg - T = ma_y$$

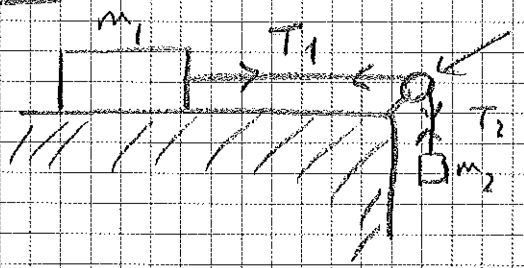
$$\Rightarrow mg - \frac{Ma_y}{2} = ma_y \Rightarrow mg = \left(m + \frac{M}{2}\right) a_y$$

$$\Rightarrow a_y = \frac{2mg}{2m+M} = \boxed{\frac{g}{1+\frac{M}{2m}}}$$

Motion with constant acceleration.

Ex 10.4:

Thin cylindrical shell with mass M and radius R



$I_{shell} = MR^2$, If there is no friction what is the angular acceleration of the pulley and T_1, T_2?

On the cylinder:

$$\sum \tau_z = I_{shell}\alpha_z \Rightarrow (T_2 - T_1)R = MR^2 \alpha_z$$
$$= MR^2 \frac{a}{R}$$
$$\Rightarrow \boxed{(T_2 - T_1) = Ma}$$

Three equations three unknowns a, T_1, T_2.

$$\left.\begin{aligned} m_2 g - T_2 &= m_2 a \\ T_1 &= m_1 a \end{aligned}\right\}$$

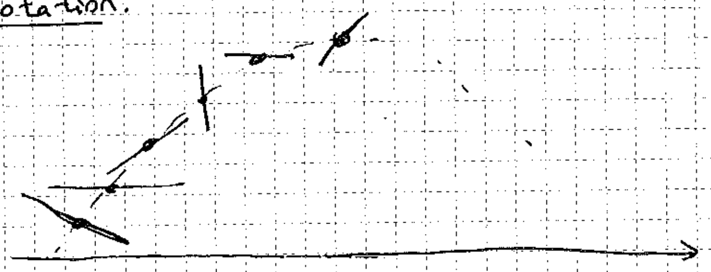
$$\Rightarrow m_2(g - a) - m_1 a = Ma$$
$$\Rightarrow m_2 g = (M + m_1 + m_2) a \Rightarrow a = \left(\frac{m_2}{M + m_1 + m_2}\right) g$$
$$\Rightarrow \alpha_z = \frac{a}{R} = \left(\frac{m_2}{M + m_1 + m_2}\right)\frac{g}{R}$$

$$\Rightarrow T_1 = m_1 a = \frac{m_1 m_2}{M + m_1 + m_2} g$$

$$T_2 = m_2(g - a) = m_2\left(g - \frac{m_2 g}{M + m_1 + m_2}\right) = \frac{m_2 (M + m_1)}{(m_1 + m_2 + M)} g$$

10.3. Rigid-Body Rotation About a Moving Axis:

What happens when the motion of a body is combined translation and rotation.

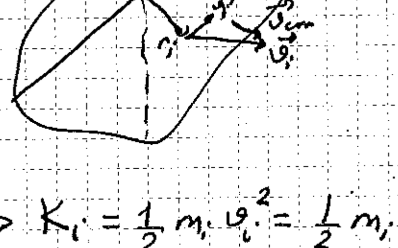


Total motion that includes translation and rotation can be represented as a combination of translational motion of the center of mass and rotation about an axis through the center of mass.

This will not be proven

$$\boxed{K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2}$$

, Body's total kinetic energy.



Consider the rigid body to be made up of particles.

$\Rightarrow$ Velocity of i^{th} particle:

$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$$

velocity of the particle relative to the center of mass

$$\Rightarrow K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i)$$

$$= \frac{1}{2} m_i (\vec{v}_{cm} + \vec{v}_i') \cdot (\vec{v}_{cm} + \vec{v}_i') = \frac{1}{2} m_i (\vec{v}_{cm} \cdot \vec{v}_{cm} + 2 \vec{v}_{cm} \cdot \vec{v}_i' + \vec{v}_i' \cdot \vec{v}_i')$$

$$= \frac{1}{2} m_i (v_{cm}^2 + 2 \vec{v}_{cm} \cdot \vec{v}_i' + v_i'^2)$$

⇒ Total kinetic energy:

$$K = \sum_i K_i = \sum_i \left(\frac{1}{2} m_i v_{cm}^2\right) + \sum_i m_i (\vec{v}_{cm} \cdot \vec{v}_i') + \sum_i \left(\frac{1}{2} m_i v_i'^2\right)$$

$$= \frac{1}{2}\underbrace{\left(\sum_i m_i\right)}_{M\text{: total mass}} v_{cm}^2 + \vec{v}_{cm} \cdot \underbrace{\left(\sum_i m_i \vec{v}_i'\right)}_{M\vec{v}_{cm}'} + \sum_i \left(\frac{1}{2} m_i v_i'^2\right)$$

$M\vec{v}_{cm}'$, from chapter 8

velocity of the center of mass relative to the center of mass ⇒ $\vec{v}_{cm}' = \vec{0}$

$$\Rightarrow K = \frac{1}{2} M v_{cm}^2 + \sum_i \left(\frac{1}{2} m_i v_i'^2\right) = \frac{1}{2} M v_{cm}^2 + \frac{1}{2}\underbrace{\left(\sum_i m_i r_i^2\right)}_{I_{cm}} \omega^2$$

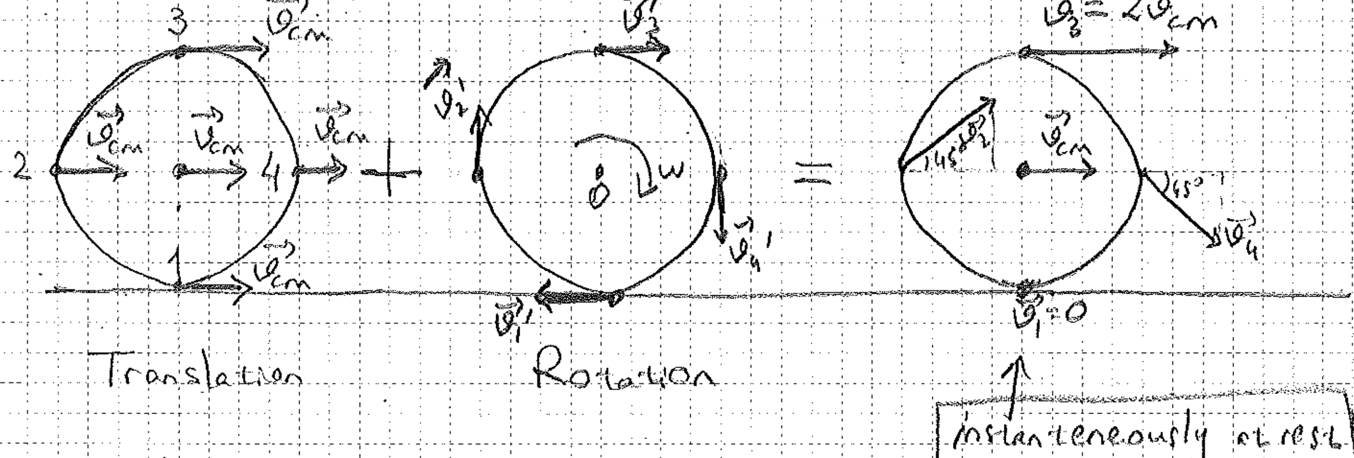
$v_i' = r_i \omega$

$$K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

moment of inertia with respect to the axis through the center of mass.

Now, we analyze the motion:

Rolling Without Slipping:



$$\vec{v}_1 = 0 \Rightarrow \vec{v}_1' = -\vec{v}_{cm} \Rightarrow \boxed{v_{cm} = R\omega = |v_1'|}$$

$|\vec{v}_1'| = |\vec{v}_2'| = |\vec{v}_3'| = |\vec{v}_4'| = R\omega$ ⇐ Rotational motion with constant angular velocity.

Kinetic energy of the wheel: $K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v_{cm}^2$

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Ex 105:

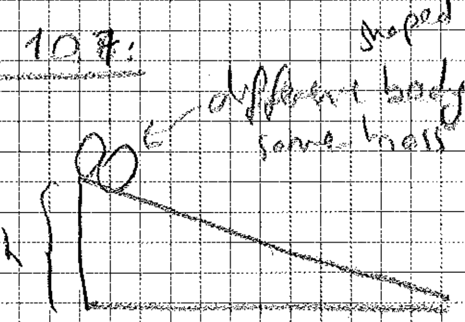
A hollow cylindrical shell with mass M and radius R rolls without slipping with speed v_{cm} on a flat surface, what is its kinetic energy?

$$K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I \omega^2, \quad I = MR^2$$

$$v_{cm} = \omega R$$

$$= \frac{1}{2} M v_{cm}^2 + \frac{1}{2} MR^2 \frac{v_{cm}^2}{R^2} = \boxed{M v_{cm}^2}$$

Ex 107:



which body will roll down faster?

$$Mgh = \frac{1}{2} I \omega^2 + \frac{1}{2} M v_{cm}^2$$

$$Mgh = \frac{1}{2} I \frac{v_{cm}^2}{R^2} + \frac{1}{2} M v_{cm}^2$$

$$\Rightarrow v_{cm} = \sqrt{\frac{2Mgh}{M + \frac{I}{R^2}}}$$

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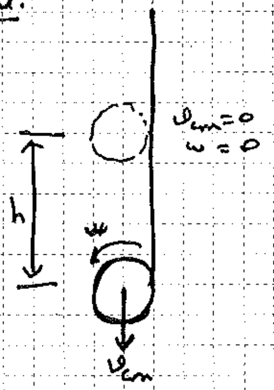
$\rightarrow$ If $I \nearrow$ $v_{cm} \searrow$

Ex 10.6:



The string unwinds without slipping or stretching. What is the speed of the center of mass, v_{cm}, after yo-yo has dropped a distance h.

⇒ All applied forces are conservative:

$$K_1 + U_1 = K_2 + U_2$$

$$\Rightarrow Mgy = \frac{1}{2} I \omega^2 + \frac{1}{2} M v_{cm}^2 \quad , \quad I = \frac{1}{2} MR^2 : \text{cylinder}$$

$$v_{cm} = R\omega$$

$$\Rightarrow \cancel{M} gy = \frac{1}{2} \frac{\cancel{M}\cancel{R^2}}{2} \frac{v_{cm}^2}{\cancel{R^2}} + \frac{1}{2} \cancel{M} v_{cm}^2$$

$$\Rightarrow gy = \frac{v_{cm}^2}{4} + \frac{v_{cm}^2}{2} \Rightarrow \boxed{v_{cm} = \sqrt{\frac{4}{3} gh}}$$

[redacted] Ex. 10.7: Rear side

Dynamics of Combined Rotation and Translation:

From chap. 8: $\sum \vec{F}_{ext} = M \vec{a}_{cm}$

From sec. 10.2: $\sum \tau_z = I_{cm} \alpha_z$

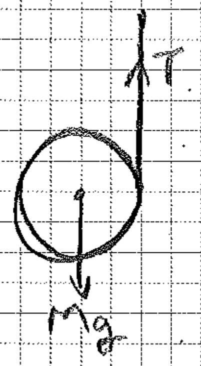
These equations can be used in analyzing the dynamics of bodies making combined rotation and translation.

$\sum \tau_z = I \alpha_z$: this equation is valid for any stationary axis of rotation.

$\sum \tau_z = I_{cm} \alpha_z$ is valid even when the axis of rotation moves.

Provided:
- The axis through the center of mass is an axis of symmetry
- The axis does not change direction.

Ex 10.8:



Find the downward acceleration of the cylinder and the tension in the string?

$$\sum F_{ext} = M a_{cm} \quad , \quad \sum \tau = I_{cm} \alpha_z$$

$$I_{cm} = \frac{1}{2} MR^2$$

$$Mg - T = M a_{cm} \quad , \quad TR = \frac{1}{2} MR^2 \alpha_z$$

$$v_{cm} = R\omega_z \quad , \quad a_{cm} = R\alpha_z$$

$$T = \frac{1}{2} M \cancel{R} \frac{a_{cm}}{\cancel{R}} = \frac{1}{2} M a_{cm}$$

$$\Rightarrow Mg - \frac{1}{2} M a_{cm} = M a_{cm} \Rightarrow \boxed{a_{cm} = \frac{2}{3} g}$$

$$T = \frac{1}{2} M a_{cm} = \boxed{\frac{Mg}{3}}$$

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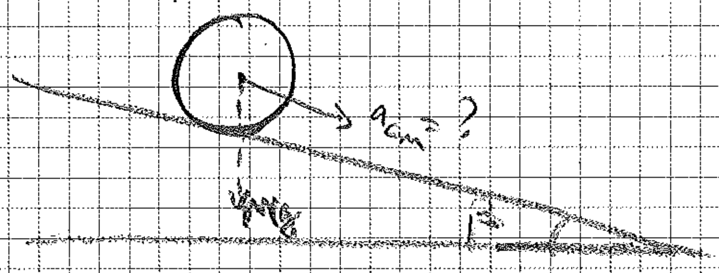
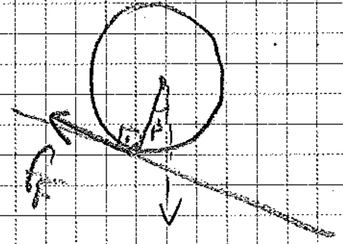
Ex 10.9. A ball rolls on a ramp inclined at an angle β to the horizontal without friction.
What is the ball's acceleration.



$$Mg\sin\beta - f_f = Ma_{cm}$$



$$f_f R = I\alpha = \left(\frac{2}{5}M\cancel{R^2}\right)\cdot\frac{a_{cm}}{\cancel{R}} = f_f \cancel{R}$$

$$\Rightarrow \frac{2}{5}Ma_{cm} = f_f$$

$$\Rightarrow \cancel{M}g\sin\beta - \frac{2}{5}\cancel{M}a_{cm} = \cancel{M}a_{cm} \Rightarrow \boxed{a_{cm} = \frac{g\sin\beta}{1+\frac{2}{5}} < g\sin\beta}$$

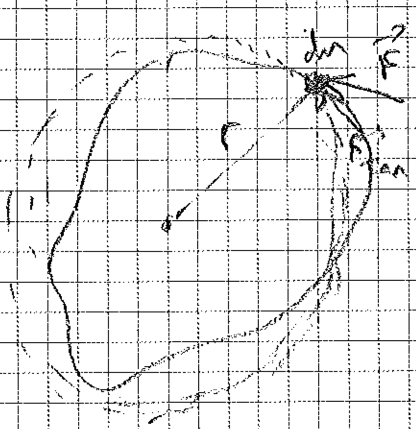
$$= \frac{5}{7}g\sin\beta$$

$$\Rightarrow \boxed{f_f = \frac{2}{7}Mg\sin\beta}$$

Rolling Friction:

If the rolling body and the surface over which it rolls are not perfectly rigid, there is a rolling friction.

10.4. Work and Power in Rotational Motion:



Consider a force $\vec{F}$ is applied to the small mass element on a rigid body.

Since dm moves along a circle the component of $\vec{F}$ along the direction of rotation is $\vec{F}_{tan}$.

$$\Rightarrow \quad dW = \vec{F}_{tan} \cdot d\vec{s} = F_{tan}\, ds = \underbrace{F_{tan}\, r}_{\tau_z}\, d\theta = \tau_z\, d\theta$$

$$\Rightarrow \boxed{W = \int dW = \int_{\theta_1}^{\theta_2} \tau_z\, d\theta}$$

Work done by $\vec{F}$ during an angular displacement from θ_1 to θ_2.

If torque is constant: $\boxed{W = \tau_z \Delta\theta}$

When a torque does work on a rotating rigid body, its rotational kinetic energy should change by an amount equal to the work done.

$$dW = \tau_z\, d\theta = I\alpha\, d\theta = I \frac{d\omega_z}{dt} d\theta = I \frac{d\theta}{dt} d\omega_z$$

$$= I\, \omega_z\, d\omega_z$$

$$\Rightarrow W = \int_{\theta_1}^{\theta_2} dW = \int I\omega_z\, d\omega_z = I \frac{\omega_z^2}{2} \Bigg|_{\omega_1}^{\omega_2} = \frac{I\omega_2^2}{2} - \frac{I\omega_1^2}{2}$$

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$\therefore$ The change in the rotational kinetic energy of a rigid body equals the work done by forces applied from outside the body.

Power associated with rotational motion:

$$P = \frac{dW}{dt} = \frac{\tau_z\, d\theta}{dt} = \boxed{\tau_z \omega_z}$$

Example 10.11:

An electric motor applies a constant torque of 10 Nm. The moment of inertia of the rotating body is 2 kg m². If the system starts from rest, find the work done by the motor in 8 s, kinetic energy at the end of this time, average power delivered by the motor?

$\tau_z = 10\,Nm, \quad I = 2\,kg\,m^2$

$\tau_z = I\alpha = \text{constant} \Rightarrow \alpha = 5\ rad/sec^2 \Rightarrow \omega(t) = 5t$

$\Rightarrow \theta(0) = 0 \Rightarrow \theta(t) = \frac{1}{2}\alpha t^2 = \frac{1}{2}\, 5 \cdot 64 = 160\ rad$

$\Rightarrow W = \tau_z \Delta\theta = 10\,Nm \times 160\,rad = \boxed{1600\,J}$

$\Delta K = \frac{1}{2} I \omega^2 = \frac{1}{2}\, 2 \times (40)^2 = \boxed{1600\,J}$

$P_{av} = \frac{W_f - W_i}{\Delta t} = \frac{1600\,J - 0\,J}{8s - 0s} = \boxed{200\,W}$

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Prob. 10.29: A 1.5 kg wheel is in the form of a solid cylinder with radius 0.1 m.

a) What const. torque will bring it from rest to an $\omega = 1200$ rev/min in 2.5 s?

b) Through what angle has it turned during that time?

c) Work done by the torque?

d) What is the wheel's kinetic energy when it is rotating at 1200 rev/min? Compare with c.

a) $\Sigma \tau_z = I\alpha$, $\alpha = \frac{\Delta\omega}{\Delta t} = \frac{1200 \text{ rev/min}}{2.5 s} = 1200 \frac{rev}{min} \frac{2\pi rad}{1 rev} \frac{1 min}{60 sec} \cdot \frac{1}{2.5 s}$

$= \frac{1200 \times 2\pi}{60 \times 2.5}$ rad/s

$\Rightarrow \tau_z = \frac{1}{2} 1.5 kg \times (0.1 m)^2 \times \frac{2\pi \times 1200}{60 \times 2.5} = \frac{\pi \times 12 \times 1.5}{60 \times 2.5} = \frac{0.3\pi}{2.5} = \boxed{\frac{3}{25}\pi \text{ Nm}}$

b) $\theta(t) = \frac{1}{2}\alpha t^2 = \frac{1}{2} \frac{1200 \times 2\pi}{60 \times 2.5} \times (2.5)^2 = 20\pi \times (2.5) = \boxed{50\pi \text{ rad}}$

c) $W = \tau_z \Delta\theta = \frac{3\pi}{25} \times 50\pi = \boxed{6\pi^2 J}$

d) $K_{final} = \frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{1}{2} (1.5 kg)(0.1)^2 \times \left(\frac{1200 \times 2\pi}{60}\right)^2$

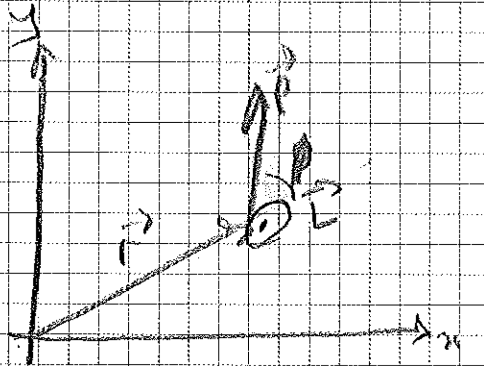
$= \frac{1}{4} \times 15 \times 10^{-3} \times (40\pi)^2 = \frac{1}{4} 15 \times 10^{-3} \times 16 \times 10^2 \pi^2 = 60 \times 10^{-1} \pi^2$

$= \boxed{6\pi^2 J}$

10.5. Angular Momentum:

In analogy with a particle's momentum in translational motion, we define the angular momentum.

Translational Motion	Rotational Motion
$\vec{F}$	$\vec{\tau} = \vec{r} \times \vec{F}$
$\vec{p}$	$\boxed{\vec{L} = \vec{r} \times \vec{p}}$: Angular momentum of a particle



$\vec{L} = \vec{r} \times \vec{p}$

$L = mvr\sin\phi$

Direction of $\vec{L}$ is given by the right-hand rule.

We know $\frac{d\vec{p}}{dt} = \vec{F}$, Newton's 2^{nd} Law.

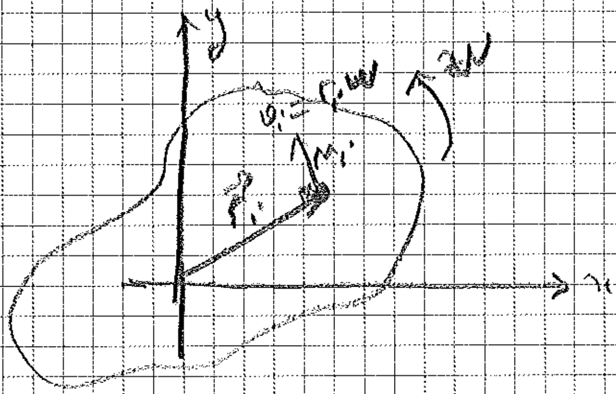
Let us analyze $\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$

$$\Rightarrow \frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v} \times (m\vec{v})}_{=0} + \vec{r} \times \vec{F} = \vec{\tau} \Rightarrow \boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

"Rate of change of angular momentum of a particle equals the torque of the net force acting on it."

Consider a thin slice of a rotating rigid body lying in the x-y plane, rotating about the z-axis.



$L_1 = m_1(r_1\omega)\, r_1 = m_1 r_1^2 \omega$

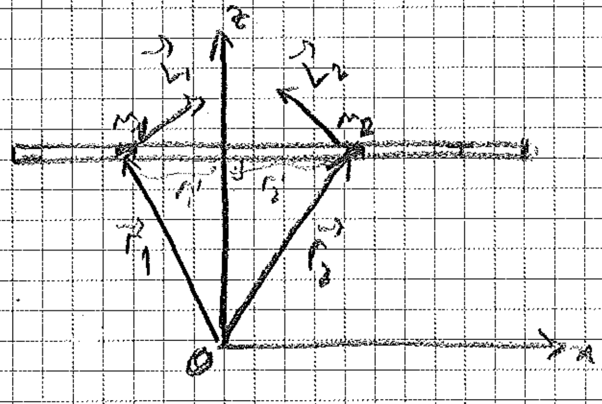
$\vec{L}_1$ is in the +z direction.

$\Rightarrow$ Total angular momentum of the slice:

$$L_z = \sum L_i = \Big(\sum_i m_i r_i^2\Big)\omega = I\omega$$

moment of inertia about the z-axis.

Now consider other slices:



$\vec{r}_2\times\vec{p}_2$, $\vec{r}_1\times\vec{p}_1$ will have components along z and along x and (y).

$$(\vec{L}_2)_z = r_2' p_2 = m_2(r_2'\omega)\, r_2' = m_2 r_2'^2 \omega$$

$$\Rightarrow \sum_i m_i r_i'^2 \omega = I\omega = L_z$$

$$\Rightarrow \boxed{L_z = I\omega_z}$$ for the whole rigid body.

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However $\vec{L}$ has components different from L_z.

In general $\vec{L} \neq I\vec{\omega}$ — L_z: angular momentum of the body. The component of $\vec{L}$ along the rotation axis.

only $L_z = I\omega_z$, z: axis of rotation.

Only if z is an axis of symmetry of the rigid body, the components of $\vec{L}$ other than z component cancel and we can write;

$$\vec{L} = I\vec{\omega} \text{ , which means } L_z = I\omega_z$$

Now let us evaluate:

For a rigid body which is made of system of particles:

$$\sum \vec{L} = \sum \vec{r_i} \times \vec{p_i} \Rightarrow \frac{d}{dt}\left(\sum \vec{r_i} \times \vec{p_i}\right) = \sum_i \vec{r_i} \times \left(\frac{d\vec{p_i}}{dt}\right)$$

$$= \sum_i \vec{r_i} \times \vec{F_i} = \sum \vec{\tau_i} = \vec{\tau} \text{ : net torque}$$

$$\Rightarrow \boxed{\underbrace{\vec{\tau}}_{\text{net torque}} = \frac{d}{dt} \underbrace{\vec{L}}_{\text{total angular momentum}}}$$

- If the system of particles rotates around a fixed axis ~~...~~

$$L_z = I\omega_z \Rightarrow \tau_z = \frac{d}{dt}(I\omega_z) = I\alpha_z$$

∴ The directions of $\vec{L}$, $\vec{\tau}$, $\vec{\omega}$ remain constant in z direc.

- If the system of particles rotates around a fixed axis which is not an axis of symmetry:

$$\tau_z = \frac{d}{dt}(I\omega_z) = I\alpha_z$$

However $\vec{\tau}_{other} = \frac{d}{dt}\vec{L}_{other}$: since $\vec{L}$ changes direction even if ω is constant, $\vec{\tau}_{other}$ is not zero.

A torque is needed to rotate at constant ω.

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Ex 10.12 :

An engine has $I = 2.5\,kg\,m^2$,
$\omega_z = 40\,t^2$

a) What is the fan's angular momentum as a function of time, and what is $L_z(t = 3s)$?

b) Net torque acting on the fan as a function of time, and $\tau_z(t = 3s) = ?$

a) $L_z(t) = I\omega_z = 2.5\,kg\,m^2 \times 40\,t^2 = \boxed{100\,t^2}$

$\boxed{L_z(3) = 900\,kg\,m^2/s}$

b) $\tau_z(t) = I\,\alpha_z = 2.5 \times 80t = \boxed{200\,t}$

$\boxed{\tau_z(3) = 600\,Nm}$

10.6. Conservation of Angular Momentum

For a rotating rigid body:

$$\underbrace{\sum \vec{\tau}}_{\text{net external torque}} = \frac{d\vec{L}}{dt}$$

$\frac{d\vec{L}}{dt}$ ↘ rate of change of the total angular momentum.

When the net external torque acting on a system is zero,

$$\frac{d\vec{L}}{dt} = 0$$: Total angular momentum of the system is conserved.

When a system consists of several parts:

$\vec{F}_{B \text{ on } A}$ $\vec{F}_{A \text{ on } B}$ B

A

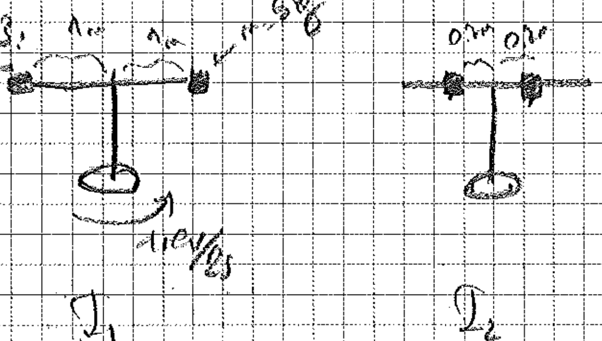
$$\vec{\tau}_{B \text{ on } A} = \frac{d\vec{L}_A}{dt} \qquad \vec{\tau}_{A \text{ on } B} = \frac{d\vec{L}_B}{dt} = -\vec{\tau}_{B \text{ on } A}$$

⇒ If the net external torque is zero:

$$\sum \vec{\tau} = \vec{\tau}_{A \text{ on } B} + \vec{\tau}_{B \text{ on } A} = 0 = \frac{d\vec{L}_A}{dt} + \frac{d\vec{L}_B}{dt} = \frac{d}{dt}(\vec{L})$$

Conservation of angular momentum

Ex 10.13:



What is the final angular velocity?
What is the effect on kinetic energy?
(ignore the moment of inertia of the body)

Since there is no net external torque:

$\vec{L}$ = constant ⇒ $L_z = I_1 \omega_{z1} = I_2 \omega_{z2}$

$$\omega_{z1} = \frac{2\pi}{2s} = \pi \text{ rad/s}$$

$\Rightarrow \quad I_1 = 2(5\,kg\,1m^2) \quad , \quad I_2 = 2 \times 5\,kg \times (0.2m)^2$

$\Rightarrow \cancel{10} \times \pi = \cancel{10} \times 0.04 \times \omega_2 \Rightarrow \quad \omega_2 = \dfrac{\pi}{0.04} = \boxed{25\pi}$

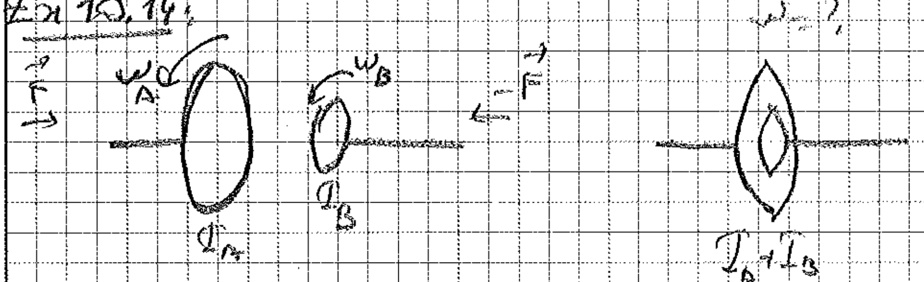
$\Rightarrow$ 12.5 revolutions/sec.

$K_1 = \frac{1}{2} I_1 \omega_1^2 = \frac{1}{2} \times 10 \times \pi^2 = 5\pi^2 \approx 45\,J$

$K_2 = \frac{1}{2} 10 \times 0.04 \times (25)^2 \pi^2 = 0.2 \times 625\pi^2 = 125\pi^2 \approx 1125\,J$

Rotational Kinetic energy increased.

Ex 10.14:

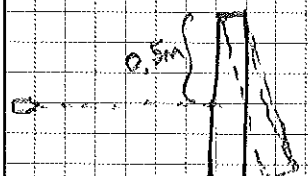


The disks are pushed towards each other such that no external torque is applied. What is the final angular velocity if they always rotate about their axis of symmetry?

$I_A \omega_A + I_B \omega_B = (I_A + I_B)\omega \Rightarrow \quad \boxed{\omega = \dfrac{I_A \omega_A + I_B \omega_B}{I_A + I_B}}$

131.5

Ex 10.16: A door 1 m wide of mass 15 kg can rotate without friction about a vertical axis. A bullet with a mass of 10 g and a speed of 400 m/s hits the center of the door, in a direction perpendicular to the door.



$$I_{door} = \frac{Md^2}{3} = \frac{15\,kg\,(1\,m)^2}{3} = 5\,kgm^2$$

Angular speed of the door just after the bullet hits the door?
Is kinetic energy conserved?

We should apply the conservation of angular momentum:

$$I_{bullet}\,\omega_{bullet} + I_{door}\,\omega_{door} = I_{total}\,\omega_{total} \quad \text{(fixed axis rotation)}$$

$$\omega_{door} = 0, \quad \omega_{bullet} = \frac{v}{0.5\,m}, \quad I_{bullet} = 10\times10^{-3}\times(0.5)^2$$

$$\Rightarrow L_{initial} = 10\times10^{-3}\times(0.5)^2\times\frac{v}{0.5} = 5\times10^{-3}\times 400 = 2\ kg\,m^2/s$$

$$2 = \left(5 + 10\times10^{-3}\times(0.5)^2\right)\omega_{total}$$

$$= (5 + 0.0025)\,\omega_{total} \Rightarrow \omega_{total} \approx 0.4\ rad/s$$

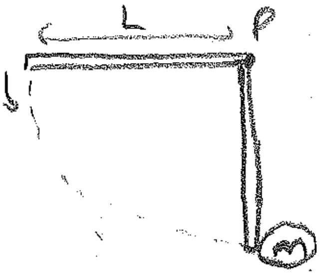
$$K_1 = \frac{1}{2}m_{bullet}v^2 = \frac{1}{2}\times10\times10^{-3}\times(400)^2 = 800\,J$$

$$K_2 = \frac{1}{2}I_{tot}\,\omega_{total}^2 = \frac{1}{2}\times5\times(0.4)^2 = 0.4\,J$$

"Section 10.7 is not included to the final"

Final Exam 2006 4 131,6

A thin, uniform rod of Length L, mass M attached to one end of a freely rotating pivot P. $I_{cm} = \frac{ML^2}{12}$, $I_p = \frac{ML^2}{3}$



(a) Ang. acc. of rod immediately after it is released?

(b) Use work-energy theo. to find the ang. vel. of the rod immediately before the contact with the ball

(c) Is the ang. momentum with respect to P conserved through the coll.?

(d) Ang. velo of rod+ball immediately after the contact?

(a) $\Sigma \tau_z = I\alpha_z \Rightarrow \cancel{M}g\frac{\cancel{L}}{2} = \frac{\cancel{M}L^{\cancel{2}}}{3}\alpha_z \Rightarrow \boxed{\alpha_z = \frac{3}{2}\frac{g}{L}}$

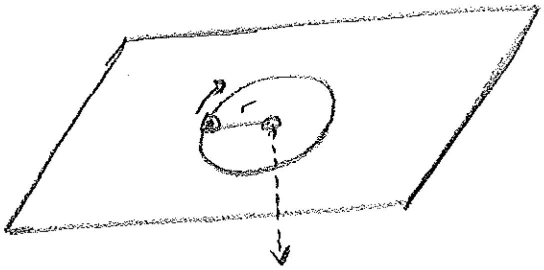
(b) $\frac{1}{2}I_p\omega_z^2 = M\frac{L}{2}g \Rightarrow \frac{1}{2}\cancel{M}\frac{L^{\cancel{2}}}{3}\omega_z^2 = \cancel{M}\frac{\cancel{L}}{2}g \Rightarrow \omega_z^2 = \frac{3g}{L} \Rightarrow \boxed{\omega_z = \sqrt{\frac{3g}{L}}}$

(c) YES, Net external torque $= 0$.

(d) $I_{rod}\,\omega_{rod} + I_m\omega_m = I_{total}\,\omega_{total}$

$\Rightarrow \frac{ML^2}{3}\cdot\sqrt{\frac{3g}{L}} = \left(\frac{ML^2}{3} + mL^2\right)\omega_{total} \Rightarrow \boxed{\omega_{total} = \left(\frac{1}{1+\frac{3m}{M}}\right)\cdot\sqrt{\frac{3g}{L}}}$

Prob. 10.39



initially $r = 0.3m$, $w_1 = 1.75$ rad/sec , $m = 0.025$ kg

cord is pulled so that $r' = 0.15m$.

(a) Is ang. momentum conserved? why?

(b) What is the new ang. speed?

(Find the change in kinetic energy of the block?

(d) How much work was done pulling the cord?

(a) Yes. Net torque due to tension is 0.

(b) $I_1 w_1 = I_1 w_2$

$\cancel{m} r_1^2 w_1 = \cancel{m} r_2^2 w_2 \Rightarrow \quad w_2 = \left(\frac{0.3}{0.15}\right)^2 (1.75) = \boxed{7 \text{ rad/s}}$

(c) $\frac{1}{2} m r_2^2 w_2^2 - \frac{1}{2} m r_1^2 w_1^2 = \frac{1}{2} m \left(\underbrace{(0.15)^2 \times 7^2}_{1.10} - \underbrace{(0.3)^2 \times (1.75)^2}_{0.28} \right)$

$= \frac{1}{2} m \times 0.82 = \boxed{0.01 \text{ J}}$

(d) Work energy theorem: $\boxed{W = 0.01 \text{ J} = \Delta K}$

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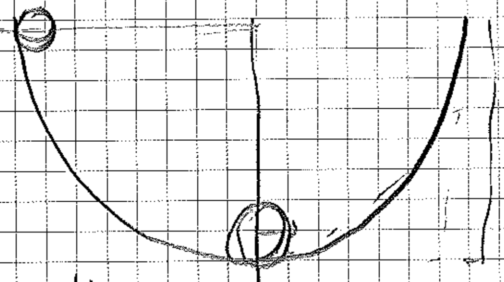
10.24:



(a) How far up the smooth side will the marble go?

(b) How far would the marble go if both sides were as rough as the left side

(c) How do you account for the fact that the marble goes higher with friction?

(a) $Mgh = \frac{1}{2} I\omega^2 + \frac{1}{2} M v_{cm}^2 = \left(\frac{1}{2} \frac{I}{R^2} + \frac{1}{2} M\right) v_{cm}^2 = Mgh$

$Mgh' + \frac{1}{2} I\omega^2 = Mgh$

$\rightarrow Mgh' = Mgh - \frac{1}{2} I\omega^2 = \frac{1}{2} M v_{cm}^2 \Rightarrow h' = \sqrt{\frac{v_{cm}^2}{2g}} < h$

(b) Marble would go up to h.

(c) rotational kinetic energy is not 0.

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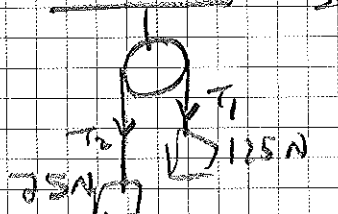
(131.91)

10.67

(disk shaped)
50 N, frictionless pulley, 0.3 m radius.
Force ceiling exerts on the hook?



$125 - T_1 = 12.5a$

$T_2 - 75 = 7.5a$

$(T_1 - T_2)R = \frac{1}{2} mR^2 \cdot \frac{a}{R} = 2.5a$

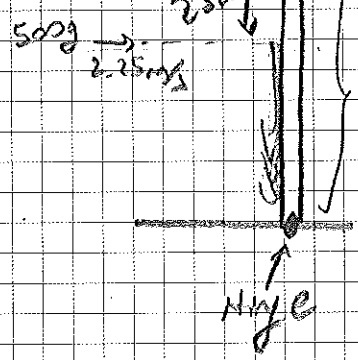
$\Rightarrow$ $125 - T_2 = 15a$, $T_2 - 75 = 7.5a$ $\Rightarrow 50 = 22.5a \Rightarrow a = \frac{50}{22.5} = 2.22$

$\Rightarrow$ $T_1 = 125 - 12.5a$ (97.25 N), $T_2 = 75 + 7.5a$ (91.65)

Force $= T_1 + T_2 + 50$ N

$= \boxed{188.9 \text{ N}} + 50$ N

10.91:



a) Ang. vel. of the bar just after it is hit

b) just as it reaches the ground

$I = \frac{1}{3} mL^2$

a) $L_{init} = 0.5 \times (0.5)^2 \times \left(\frac{2.25}{0.5}\right) = 0.5625 \ \frac{kgm^2}{s}$

b) just after the coll.: $I = 0.28125 \ kgm^2$

$0.5625 = \frac{1}{3}(1.5)(0.75)^2 \times w \Rightarrow w_i = 2$ rad/s

$\frac{1}{2} I w_i^2 + mgh = \underbrace{\frac{1}{2} \times 0.28125 \times 4}_{0.5625} + \underbrace{15 \times 0.375}_{5.625} = \frac{1}{2} 0.28125 w_f^2 \Rightarrow w_f = 6.6$ rad/s

$\Rightarrow L_f = I w_f = 0.5625 \times 6.6 = \boxed{3.7} \ \frac{kgm^2}{s}$

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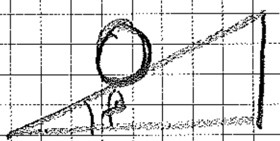
131,9

10.26, A bowling ball rolls without slipping up a ramp. Treat the ball as a uniform solid sphere.

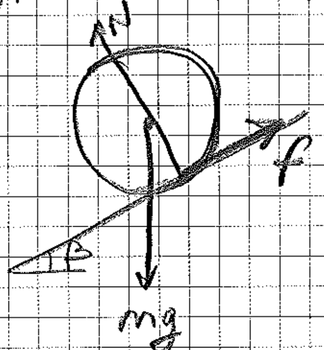


(a) Draw the free body diagram

(b) Acceleration of the center of mass of the ball?

(c) Minimum coefficient of static friction to prevent slipping?

(a)



f uphill because it generates the necessary torque

(b) $mg\sin\beta - f = ma$, $f\cancel{R} = \frac{2}{5}mR^{\cancel{2}}\frac{a}{\cancel{R}} \Rightarrow \boxed{f = \frac{2}{5}ma}$

$\cancel{m}g\sin\beta = \frac{7}{5}\cancel{m}a \Rightarrow \boxed{a = \frac{5}{7}g\sin\beta}$

(c) $f \leq \mu_s mg\cos\beta \Rightarrow$ at min. μ_s $\frac{2}{5}\cancel{m}\frac{5}{7}\cancel{g}\sin\beta = \mu_s \cancel{m}\cancel{g}\cos\beta$

$\Rightarrow \boxed{\mu_s = \frac{2}{7}\tan\beta}$