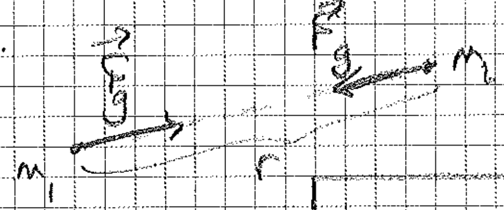
Chapter 13.

Gravitation:

13.1 Newton's Law of Gravitation

There is a gravitational attraction between any two bodies:



$$\boxed{F_g = G\frac{m_1 m_2}{r^2}}$$ Law of Gravitation

G → gravitational constant $= 6{,}67\times10^{-11}\ Nm^2/kg^2$

for example $m_1 = 60kg$, $m_2 = 60kg$, $r = 1m$

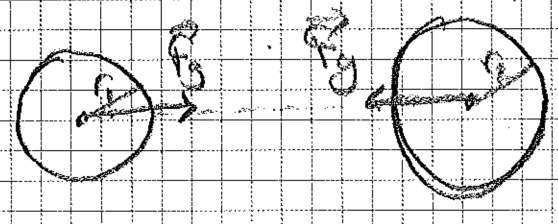
$\Rightarrow F_g = 6{,}67\times10^{-11}\times36\times10^2 \approx 250\times10^{-9} = \boxed{25\times10^{-8}\ N}$ very small!

- F_g acts along the line joining the two particles. Gravitational forces acting on the bodies form an action-reaction pair:

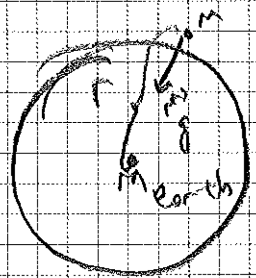
$$\vec{F}_{1 on 2} = -\vec{F}_{2 on 1}$$

- Gravitational force is often used in analyzing planetary motion.
It is important to note that when the particles have spherical symmetry;

$F_g = G\frac{m_1 m_2}{r^2}$ is valid even if the distance between the particles is comparable to their size



We call the Earth's gravitational attraction as weight:



$$F_g = G\,\frac{M_{earth}\, m_1}{r^2}$$

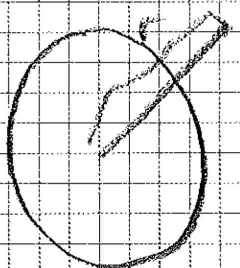
$r = R_{earth} + y \approx R_{earth}$ (At a point on the Earth's surface)

$$\Rightarrow \boxed{F_g \approx g\, m_1}, \quad g = G\,\frac{M_{earth}}{R_{earth}^2}$$

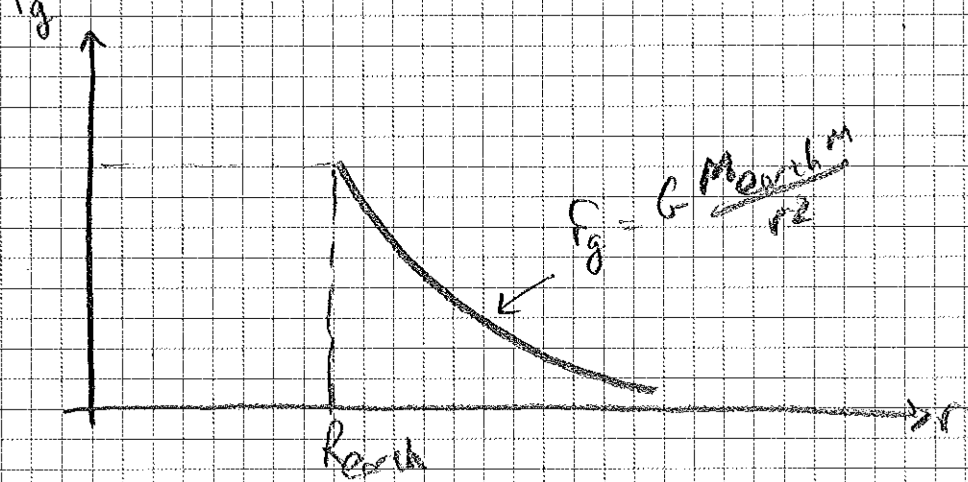
Given that $R_E = 6380\,km$, $M_{earth} = 5,94 \times 10^{24}\,kg$

$$g = 6.67\times10^{-11} \times \frac{5,94\times10^{24}}{(6380\times10^{3})^2} = 9,8\, m/s^2$$

At a point above the Earth's surface:



$$\underset{\text{weight}}{W} = F_g = G\,\frac{M_{earth}\, m}{r^2}$$



Ex 12.4: Mars has a radius of $r_m = 3.4\times10^{6}\,m$, and $M_m = 6.42\times10^{23}\,kg$

If a person weighs $60\,kg \times 9.8\,m/s^2 \approx 600\,N$ on Earth

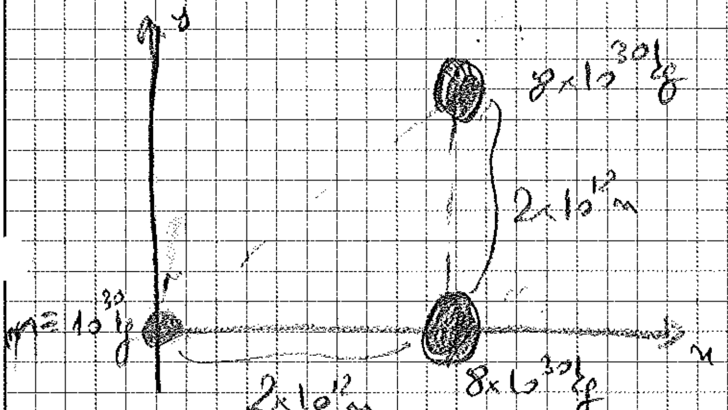
How much will he/she weigh on Mars?

$$W_{Mars} = G\frac{M_m m}{r_m^2} = 6.67\times10^{-11}\times\frac{(6.42\times10^{23})\times60}{(3.4\times10^{6})^2}$$

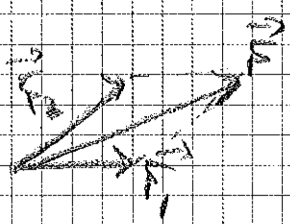
$$= \frac{6.67\times6.42}{(3.4)^2}\times\frac{10^{12}}{10^{12}}\times60 \approx \frac{7\times6}{(3.4)^2}\times60$$

$$\approx \frac{42\times60}{12} = \frac{25.2}{12}\times10 \approx \boxed{210\,N}$$

Ex 12.3:



Three spherical stars are located in the configuration shown. What is the magnitude and direction of the total gravitational force on the small star?



$$\vec{F}_1 = 6.67\times10^{-11}\times\frac{8\times10^{30}\times10^{30}}{(2\times10^{12})^2}\,\hat{i}$$

$$= 13.3\times10^{60-11-24}\,\hat{i} = 13.3\times10^{25}\,N\,\hat{i}$$

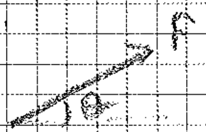
$$\vec{F}_2 = 6.67\times10^{-11}\times\frac{8\times10^{30}\times10^{30}}{(2\sqrt{2}\times10^{12})^2}\left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}\right)$$

$$= 6.67\times10^{25}\left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}\right)$$

$\Rightarrow \vec{F}_2 \approx 4.72 \times 10^{25}\,N\,\hat{i} + 4.72 \times 10^{25}\,N\,\hat{j}$

$\Rightarrow \vec{F} = \vec{F}_1 + \vec{F}_2 = 18.1 \times 10^{25}\,N\,\hat{i} + 4.72 \times 10^{25}\,N\,\hat{j}$

$$\Rightarrow \boxed{|\vec{F}| = 18.7 \times 10^{25}\,N}$$

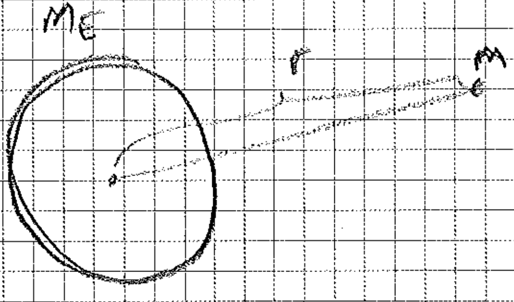


$\theta = \arctan\left(\frac{F_y}{F_x}\right) = 14.6^\circ$

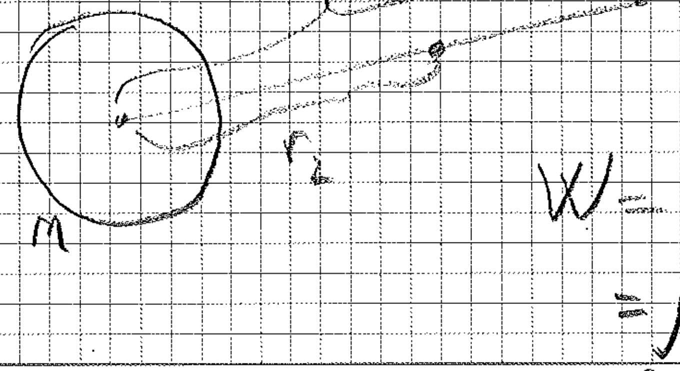
12.3. Gravitational Potential Energy:

Gravitational force is a conservative force, so there is a potential energy associated with the gravitational force.

Previously we have assumed: $U = mgy$, Potential Energy due to the Earth's gravitational force.



In general U will depend on r, the distance of the body from the Earth's center.



Consider a mass m is brought from r_1 to r_2 only under the influence of the gravitational force:

$W = U_1 - U_2$: work done

$$= \int_{r_1}^{r_2} \left(-G\frac{Mm}{r^2}\right)\hat{e}_r \cdot d\vec{r}$$

$$W = \int_{r_1}^{r_2} \left(-G\frac{Mm}{r^2}\hat{e}_r\right)\cdot d\vec{l}$$, line integral

In spherical coordinates: $d\vec{l} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\phi\,\hat{e}_\phi$

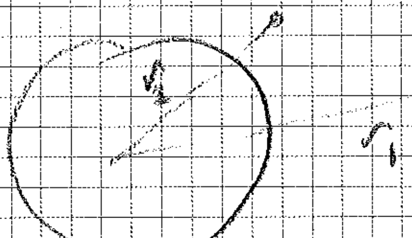
$$\Rightarrow W = \int_{r_1}^{r_2} -G\frac{Mm}{r^2}\,dr = -GMm\left(-\frac{1}{r}\Big|_{r_1}^{r_2}\right)$$

$$= \frac{GMm}{r_2} - \frac{GMm}{r_1} = U_1 - U_2$$

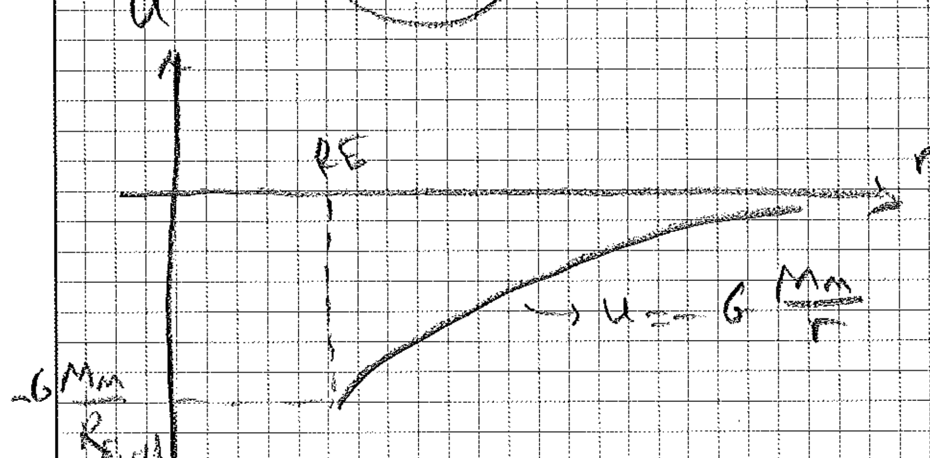
$$\Rightarrow U_1 = -G\frac{Mm}{r_1} \Rightarrow \text{We define } \boxed{U(r) = -G\frac{Mm}{r}}$$

The gravitational potential energy.

Note that the same work would be done if $\theta_1 \neq \theta_2$ or $\phi_1 \neq \phi_2$

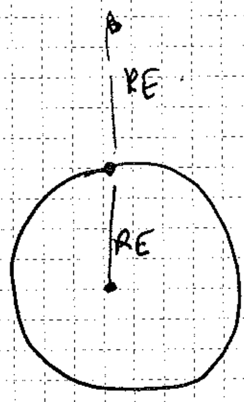


$W = U(r_1) - U(r_2)$



We assume $\lim_{r\to\infty} U(r) = 0$

Example 12.5:



A rocket is fired from the Earth's surface.

a) The initial speed necessary to shoot the rocket to a height of R_E from the Earth's surface.

b) Escape speed? Initial speed necessary so that the rocket completely escapes from the Earth.

a) $K_1 + U_1 = K_2 + U_2$

$$\Rightarrow \frac{1}{2}\cancel{m}v_1^2 - G\frac{m_E \cancel{m}}{R_E} = 0 - G\frac{m_E \cancel{m}}{2R_E}$$

$$\Rightarrow \frac{1}{2}v_1^2 = G\frac{m_E}{2R_E} \Rightarrow v_1 = \sqrt{G\frac{m_E}{R_E}} = \boxed{7900\,m/s}$$

b) $K_1 + U_1 = K_2 + U_2$, $r_2 \to \infty$

$$\Rightarrow \frac{1}{2}\cancel{m}v_1^2 - G\frac{m_E \cancel{m}}{R_E} = 0 - 0 \Rightarrow v_1^2 = \frac{2Gm_E}{R_E}$$

$$v_1 = \sqrt{\frac{2Gm_E}{R_E}} = \boxed{11200\,m/s}$$

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Prob 12.24: When in orbit, a communication satellite attracts the earth with a force of 19 kN and the earth-satellite gravitational potential energy is -1.39×10^{11} J.

a) What is the satellite's altitude above the earth's surface?

b) What is the mass of the satellite?

a) $F_G = G\dfrac{M_e m}{r^2} = 19\,kN$

$U = -G\dfrac{M_e m}{r} = -1.39\times10^{11}\,J$

$\Rightarrow \dfrac{F_G}{|U|} = \dfrac{1}{r} = \dfrac{19\times10^3}{1.39\times10^{11}}$

$\Rightarrow r = 0.073\times10^8$

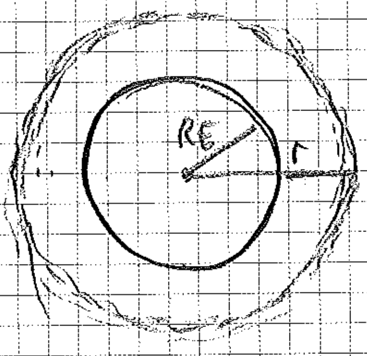
$r = 73\times10^5 = 7300\,km$

b) $U = -G\dfrac{M_e m}{r} = -1.39\times10^{11}$

$\Rightarrow 6.67\times10^{-11} \times \dfrac{5.97\times10^{24}}{73\times10^5}\, m = 1.39\times10^{11}$

$\Rightarrow m = 2.55\times10^{11+5-24+11} = 2.55\times10^3\,kg$

12.4. The Motion of Satellites:



Consider a satellite rotating around the earth in a circular orbit.

In such an orbit, the speed of the satellite is constant.

⇒ The satellite makes uniform circular motion.

- Net force applied to the satellite is:

$$\sum \vec{F} = \vec{F}_g = -G\frac{M_e m}{r^2}\hat{r} \quad : \text{along the radial direction.}$$

For uniform circular motion: $a_{rad} = \frac{v^2}{r} = G\frac{M_e}{r^2} \Rightarrow v = \sqrt{G\frac{M_e}{r}}$

⇒ Satellite's motion does not depend on its mass.

⇒ A person in a space shuttle will feel the same acceleration as the shuttle. ⇒ The person will feel weightless.

- The energy of the satellite:

$$\left.\begin{aligned} U &= -G\frac{M_e m}{r} \\ K &= \frac{1}{2}mv^2 = \frac{1}{2}mG\frac{M_e}{r} = G\frac{M_e m}{2r} \end{aligned}\right\} \quad \begin{aligned} &\text{Total Energy} \\ E &= K + U \\ &= -\frac{1}{2}G\frac{M_e m}{r} \end{aligned}$$

- The period of the circular motion:

$$vT = 2\pi r \Rightarrow T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM_e}}$$

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Ex 12.6: We want to place a 1000kg satellite into a circular orbit 300km above Earth's surface.

a) What speed, period and radial acceleration must it have?

b) How much work has to be done to place this satellite into the orbit.

c) How much additional work would have to be done to make the satellite escape the earth?

a) $\frac{v^2}{r} = G\frac{M_e}{r^2} \Rightarrow v = \sqrt{G\frac{M_e}{r}} = \sqrt{\frac{6.67\times10^{-11} \times 5.97\times10^{24}}{6680\times10^3 m}} = 7720 m/s$

$r = 6380 + 300 = 6680 km$

$T = \frac{2\pi r}{v} = \frac{2\pi \times 6680\times10^3}{7720 m/s} = 5440 s = 90.6 min.$

$a_{rad} = \frac{v^2}{r} = \frac{(7720)^2}{(6680\times10^3)} = 8.92 m/s^2$

b) Originally the Satellite is located on earth:

$E_1 = \underbrace{K_1}_{=0} + U_1 = -G\frac{M_e m}{R} = -6.25\times10^{10} J$

$E_2 = K_2 + U_2 = -G\frac{M_e m}{2r} = -2.99\times10^{10} J$

$\Rightarrow W = \Delta E = E_2 - E_1 = 3.26\times10^{10} J$

"satellite initially not in orbit"

Earth's rotation $v = 460 m/s \Rightarrow K_1 = \frac{1}{2}10^3 \times (460)^2 \sim 10^8 J \ll 10^{10} J$

c) $E_2 = 0 \Rightarrow$ Additional work $= 0 + G\frac{M_e m}{2r}$

$= \frac{6.67\times10^{-11} \times 5.97\times10^{24} \times 1000}{2\times 6680\times10^3} = 2.99\times10^{10} J$

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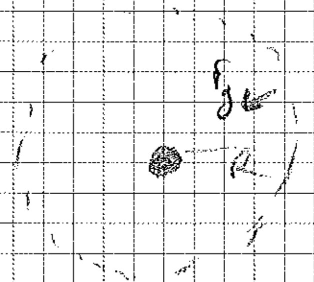
Motion of Satellites:

In a circular orbit:



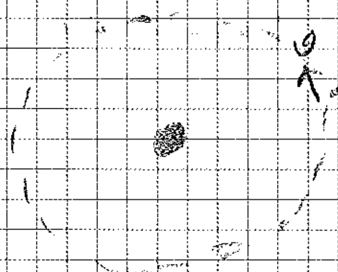
$$F_g = G\frac{m_e m}{R^2} = m\frac{v^2}{R} \Rightarrow v^2 = G\frac{m_e}{R}$$

$$\Rightarrow v = \sqrt{G\frac{m_e}{R}}$$

$$\Rightarrow U = -G\frac{m_e m}{R}, \quad K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{Gm_e m}{R} \Rightarrow E = K+U$$

$$= -G\frac{m_e m}{2R}$$

Escape Velocity of a Satellite:



We apply a velocity v_{escape} in the radial direction to the satellite:

$$\Rightarrow K = \frac{1}{2}m\left(v^2 + v_{escape}^2\right)$$

If this results in the escape of the satellite from the circular orbit:

$$E' = \frac{1}{2}m\left(v^2 + v_{escape}^2\right) - G\frac{m_e m}{R} = 0$$

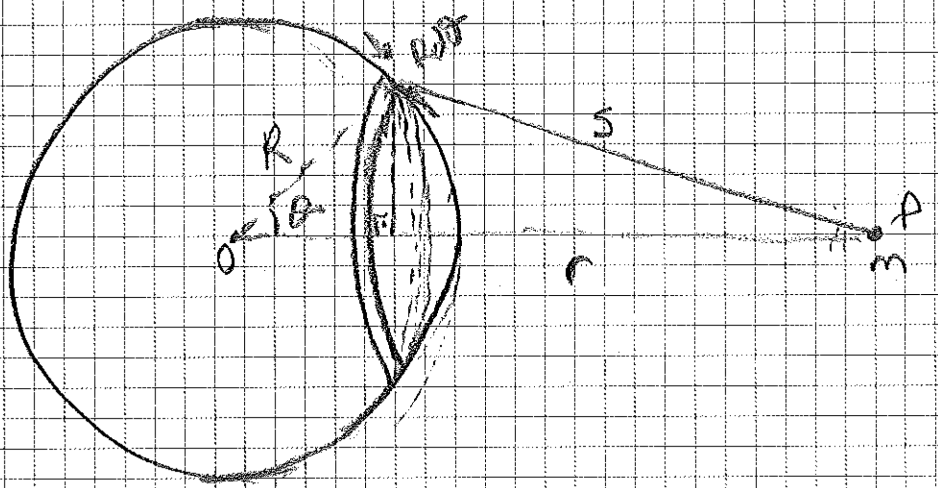
$$\Rightarrow \frac{1}{2}mv_{escape}^2 = \frac{Gm_e m}{2R} \Rightarrow v_{escape} = \sqrt{\frac{Gm_e}{R}}$$

Escape velocity of a satellite

12.6. Spherical Mass Distribution:

We will show that the gravitational interaction between two spherically symmetric mass distributions is the same as though all the mass of each were concentrated at its center.

Consider a point mass m interacting with a thin spherical shell with total mass M.



Point mass m is a distance r away from the center of the shell

Consider a ring on the spherical shell

$$dU = -G\frac{dM\, m}{s} \quad , \quad dM: \text{mass of the ring} = \frac{M}{4\pi R^2}\cdot 2\pi R\sin\theta\, R\, d\theta$$

$$= -Gm\frac{M}{4\pi R^2}\frac{2\pi R^2}{s}\sin\theta\, d\theta = -\frac{GmM}{2}\frac{\sin\theta}{s}d\theta$$

use the cosine theorem:

$$s^2 = r^2 + R^2 - 2rR\cos\theta$$

$$\Rightarrow U = -\frac{GmM}{2}\int_0^{\pi}\frac{\sin\theta\, d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}}$$

$$U = -\frac{GmM}{2}\frac{1}{\sqrt{2rR}}\int_0^{\pi}\frac{\sin\theta\, d\theta}{\sqrt{\frac{r^2+R^2}{2rR}-\cos\theta}}$$

call $\cos\theta = x \Rightarrow -\sin\theta\, d\theta = dx \Rightarrow U = \frac{GmM}{2}\frac{1}{\sqrt{2rR}}\int_1^{-1}\frac{dx}{\sqrt{\frac{r^2+R^2}{2rR}-x}}$

$$\Rightarrow U = \frac{GmM}{2}\frac{1}{\sqrt{2rR}}(-2)\left(\frac{r^2+R^2}{2rR}-x\right)^{1/2}\Bigg|_1^{-1}$$

$$= -\frac{GmM}{\sqrt{2rR}}\left\{\left(\frac{r^2+R^2}{2rR}+1\right)^{1/2}-\left(\frac{r^2+R^2}{2rR}-1\right)^{1/2}\right\}$$

$$= -\frac{GmM}{2rR}\left\{r+R-|r-R|\right\} = -G\frac{mM}{2rR}2R = -\frac{GmM}{r}$$

Same potential energy as the potential energy of two point masses m and M at a distance r.

$\Rightarrow$ Any spherically symmetric mass distribution can be thought of as a combination of concentric spherical shells.

$\Rightarrow$ Because of the principle of superposition, any spherically symmetric mass distribution can be represented as a point mass.

Gravitational Force:

$$\vec{F} = -\vec{\nabla}U = -\frac{GmM}{r^2}\hat{e}_r$$

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Point mass inside the Spherical Shell:

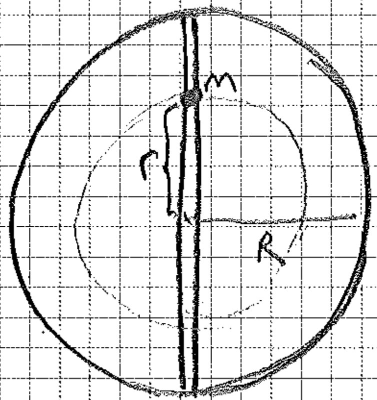
If the point P were inside the Shell, then the calculations would be the same up to:

$$U = -\frac{GmM}{2rR}\Big(r+R-\underbrace{|r-R|}_{=R-r}\Big) = -\frac{GmM}{2R} = \text{constant.}$$

$\Rightarrow \vec{F}_G = -\vec{\nabla}U = 0$, The spherical shell does not apply any force on the mass m located inside.

An intermediate Situation:

Consider a sphere with a constant mass density, and a small hole drilled in it.



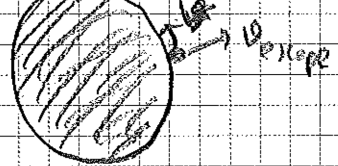
A mass m is located in the hole at a distance r from the center. What is the force on m?

The shells with radius larger than r will not apply a force to the mass.

The remaining mass will apply a force as if all mass was concentrated in the center of the sphere.

$$\Rightarrow F_G = -G\frac{m}{r^2}\frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = \boxed{-\frac{GmM}{R^3}r}$$

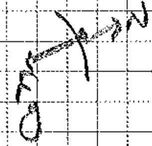
Escape velocity of a Rocket on Earth:



$$E = \frac{1}{2} m v^2 - G \frac{m_e m}{R}$$

$$E' = \frac{1}{2} m (v^2 + v_{escape}^2) - G \frac{m_e m}{R} = 0$$

$$\frac{1}{2} m v^2 = \frac{1}{2} m \left(\left(\frac{2\pi \times 6380 \times 10^3}{(24 \times 60 \times 60)} \right)^2 \right) \approx \frac{1}{2} m \left(\left(440 \frac{m}{s}\right)^2 \right)$$



$$F_g - N = \frac{m v^2}{R}$$

$$\Rightarrow G \frac{m_e m}{R^2} - N = \frac{m v^2}{R}$$

$$E' = \frac{1}{2} m (v^2 + v_{escape}^2) - G \frac{m_e m}{R} = 0$$

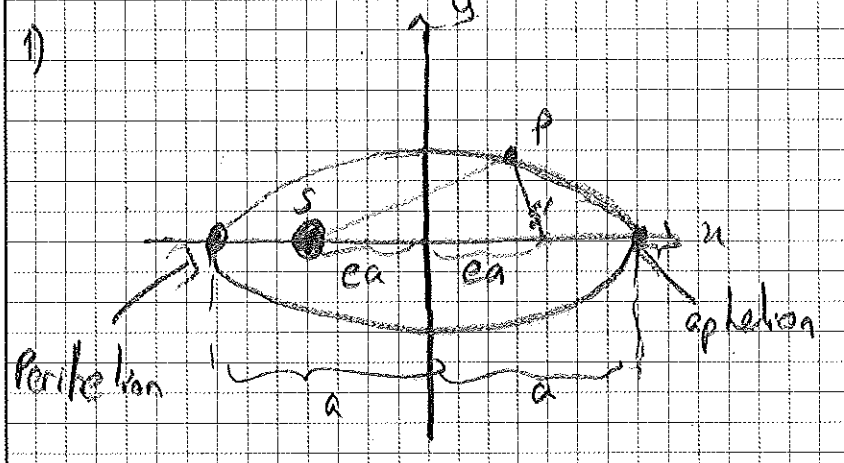
If $v_{escape} \gg v \Rightarrow v_{escape} = \sqrt{\frac{2 G m_e}{R_e}} = \boxed{11200 \, m/s}$

So, we can ignore the Earth's rotation!

12.5. Kepler's Laws and the Motion of Planets:

Kepler's Laws explain the motion of planets around the Sun. These Laws were established in the beginning of 1600s by Johannes Kepler.

1)



Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

$SP + S'P$ is constant $\Rightarrow$ ellipse.

$$SP + S'P = 2a$$

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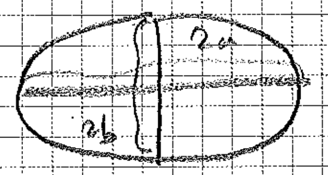


2a : major axis length of the ellipse
2b : minor axis length of the ellipse

2ea : separation between the two focus points.

e : is called as the eccentricity
a : is called as the semi-major axis.

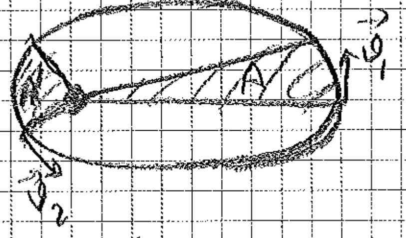
Note that $0 \leq e < 1$

If $e = 0 \Rightarrow$ Ellipse becomes a circle.

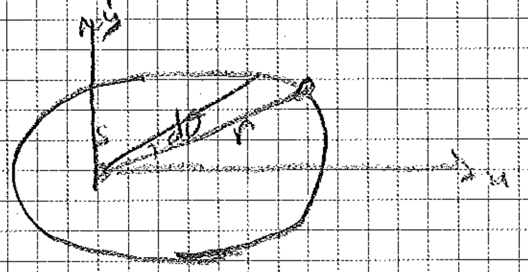
* It is possible to show that, for a body acted on by an attractive force proportional to $\frac{1}{r^2}$, the only possible closed orbits are a circle or an ellipse.

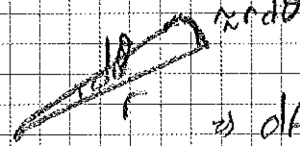
2) A line from the Sun to a given planet sweeps out equal areas in equal times.

$\frac{\Delta A}{\Delta t}$ = constant = sector velocity.



If we place the Sun to the center of the coordinate system,





$$\Rightarrow dA = \frac{1}{2} r (r d\theta) = \frac{1}{2} r^2 d\theta$$

$$\Rightarrow \boxed{\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}}$$

When r is small $\frac{d\theta}{dt}$ is large

r is large $\frac{d\theta}{dt}$ is small.

Remember the circular motion;

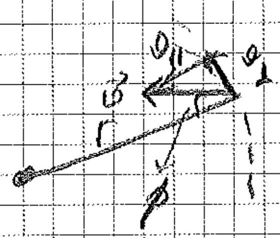
$$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \vec{v} = \left(r(-\sin\theta)\frac{d\theta}{dt}\,\hat{i} + r\cos\theta\,\frac{d\theta}{dt}\,\hat{j}\right) + \frac{dr}{dt}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$$

change of radius vector magnitude

change of ~~velocity~~ radius vector direction

$$= r\frac{d\theta}{dt}\,\hat{e}_\theta + \frac{dr}{dt}\,\hat{e}_r$$



$$v_\perp = r\frac{d\theta}{dt} = v\sin\phi$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2}\,r\,v_\perp = \boxed{\frac{1}{2}\,r\,v\sin\phi}$$

What about the angular momentum:

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times m\vec{v} \Rightarrow |\vec{L}| = m\,r\,v\sin\phi = 2m\frac{dA}{dt} = \text{constant},$$

Motion is on a plane $\Rightarrow$ direction of $\vec{L}$ is $+\hat{z}$ is constant.

$\Rightarrow$ $\vec{L}$: angular momentum remains constant.

3) The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.

For an elliptical orbit:

one can find that ;

$$\boxed{T = \frac{2\pi a^{3/2}}{\sqrt{GM_s}}}$$

The period does not depend on the eccentricity.

Ex 12.7: At what point in an elliptical orbit does a planet have the greatest speed?

$\frac{dA}{dt} = \frac{1}{2} r v \sin\phi = \text{constant}$

when planet is at the point closest to the Sun:

$\frac{1}{2} r_{min} v = \text{constant} \Rightarrow \boxed{v \text{ is maximum}}$

Another approach:

$E = \frac{1}{2} m v^2 - G\frac{M_{sun} m}{r} = \text{constant} \Rightarrow v = \sqrt{\frac{2GM_{sun}}{r}}$ is maximum

when r is minimum.

Ex 12.8: An asteroid has an orbital period of 4.62 years. What is the semi major axis of the orbit?

$a = ?$

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM_s}} \Rightarrow a^{3/2} = \frac{T\sqrt{GM_s}}{2\pi}$$

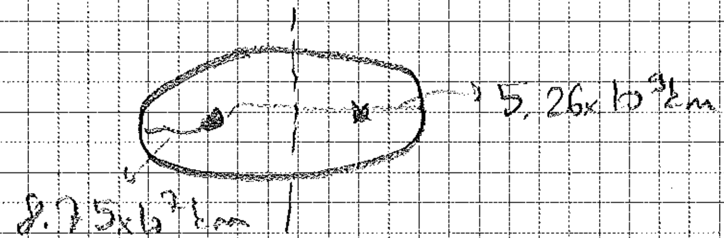
$$\Rightarrow a = \left(\frac{T\sqrt{GM_s}}{2\pi}\right)^{2/3} = \left(\frac{T^2 GM_s}{4\pi^2}\right)^{1/3}$$

$$= \left(\frac{(4.62\times 365\times 24\times 60\times 60)^2 \times (6.67\times 10^{-11})\times (1.99\times 10^{30})}{4\pi^2}\right)^{1/3}$$

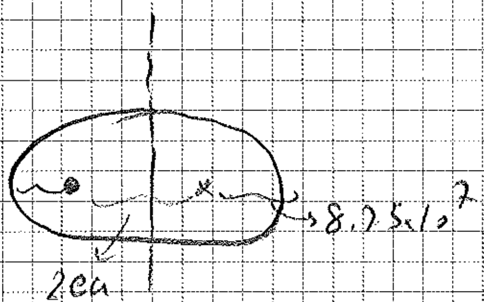
$$= 4.15\times 10^{11}\ \text{m}$$

Ex 12.9: Comet Halley moves in an elongated orbit around the sun. At perihelion, the comet is 8.75×10^{7} km from the sun, at aphelion it is 5.26×10^{9} km from the sun.

What are a, e, and T?



$$2a = 8.75\times10^{7}\times10^{3} + 5.26\times10^{9}\times10^{3} \Rightarrow \boxed{a = 2.67\times10^{9}\ \text{km}}$$



$$2ea + 8.75\times10^{7}\ \text{km} = 5.26\times10^{9}$$

$$\Rightarrow 2ea = 517\times10^{7}\ \text{km}$$

$$\Rightarrow e = \frac{517\times10^{7}}{534\times10^{7}} = \boxed{0.967}$$

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_s}} = \frac{2\pi\times(2.67\times10^{12}\ \text{m})^{3/2}}{\sqrt{6.67\times10^{-11}\times1.99\times10^{30}}} = \boxed{75.5\ \text{years}}$$

KOÇ ÜNİVERSİTESİ

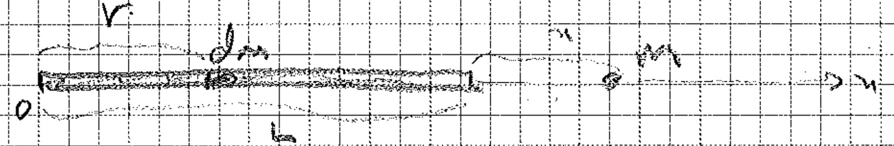
Example: Consider the Earth's orbit around the Sun to be almost circular. What is the period of the Earth's motion if the distance between the Earth and the Sun is 1.50×10^{11} m.

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_s}}$$

$$= \frac{2\pi \times (1.50\times10^{11})^{3/2}}{\sqrt{6.67\times10^{-11}\times1.99\times10^{30}}} = \boxed{31.68\times10^{6}\,\text{sec}} = 367\,\text{days} = 365.3\,\text{days}$$

13.32, 13.33, 13.34

13.32



$$dU = -G\frac{m\,dm}{(x+L-r)} = -G\frac{m\lambda\,dr}{(x+L-r)}$$

$$U = \int_0^L -\frac{Gm\lambda\,dr}{(x+L-r)} = +Gm\lambda \ln(x+L-r)\Big|_0^L$$

$$= Gm\lambda \ln\left(\frac{x}{x+L}\right)$$

when $x \gg L$: $U = Gm\lambda \ln\left(\frac{1}{1+\frac{L}{x}}\right) = -Gm\lambda \ln\left(1+\frac{L}{x}\right)$

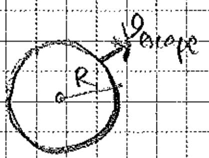
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots \Rightarrow \boxed{U \simeq -Gm\lambda\frac{L}{x}}$$

$F_g = -G\frac{Mm}{R^3}r$, is of the form $F = -kx$

$\Rightarrow$ The particle will perform oscillations.

12.8. Black Holes:

$v_{escape} = \sqrt{\frac{2GM}{R}}$ means that particles which have smaller speed than the escape speed will remain on the planet.



If the planet is so dense that M is very large and R is very small,

$v_{escape} = c$ will be reached.
$\uparrow$ speed of light

In this case, even light cannot escape from the planet.

$\Rightarrow$ Black hole
(no light can come from the planet)

$c = \sqrt{\frac{2GM}{R}} \Rightarrow \boxed{R_s = \frac{2GM}{c^2}}$ Schwarzchild radius.

Prob. 13.33:

Uniform Ring shaped object of mass M.

(a) U of the system

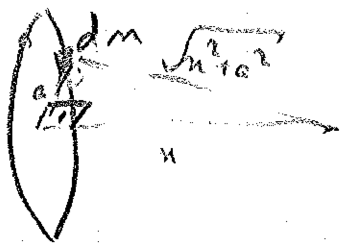
(b) Show that the answer reduces to the expected answer for $x \gg a$

(c) Use $F_x = -\frac{dU}{dx}$ to find the magn. and direction of the force.

(d) Show that the answer reduces to the expected answer for $x \gg a$

(e) Values of U and F_x for $x=0$ and how these make sense

(a)



$$dU = -G\frac{m\,dm}{\sqrt{x^2+a^2}},$$

$$\Rightarrow U = \int -\frac{Gm\,dm}{\sqrt{x^2+a^2}} = -\frac{Gm}{\sqrt{x^2+a^2}} \underbrace{\int dm}_{\text{Total mass of the ring}} = \boxed{-\frac{GmM}{\sqrt{x^2+a^2}}}$$

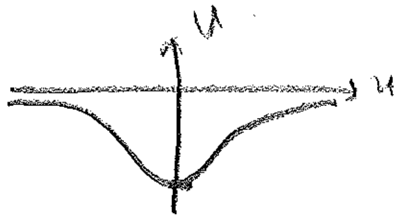
(b) $x \gg a \Rightarrow U \approx -G\frac{mM}{x}$, pot. energy of two point particles!

(c) $F_x = -\frac{d}{dx}\left(-\frac{GmM}{\sqrt{x^2+a^2}}\right) = (GmM)\,-\frac{1}{2}\,2x\,(x^2+a^2)^{-3/2}$

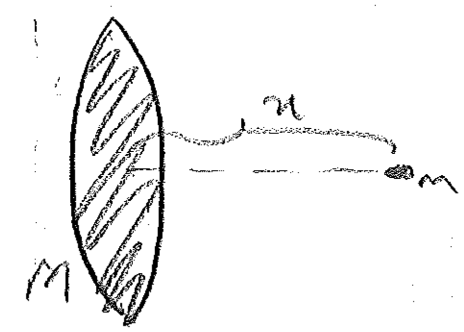
$$= \boxed{-\frac{GmM\,x}{(x^2+a^2)\sqrt{x^2+a^2}}} \quad \leftarrow \text{sign negative! (for } x>0)$$

(d) $x \gg a \Rightarrow F_x \approx -G\frac{mM}{x^2}$ as expected from 2 point particles.

(e) $\underbrace{U(0) = -G\frac{mM}{a}}_{\text{minimum pot.}}$, $\underbrace{F_x(0) = 0}_{\text{no force}}$



Prob. 13.89: Gravitational force between the disk and point particle?

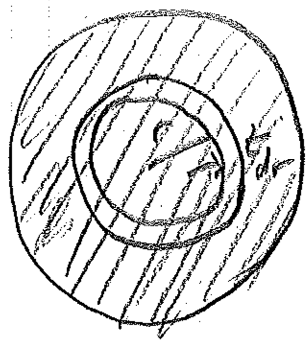


disk of radius a

For a ring:

$$F_x = -\frac{GmMx}{(x^2+r^2)^{3/2}}$$

Consider an infinitesimal ring in the disk:



radius of the ring: r
width of the ring: dr

$$\Rightarrow dF_x = -G\frac{m\,dm\,x}{(x^2+r^2)^{3/2}}$$

, dm: mass of the ring

$= 2\pi r\,dr\,\sigma$

mass density $= \frac{M}{\pi a^2}$

$$= -Gmx\frac{2\pi r\,dr\,\sigma}{(x^2+r^2)^{3/2}}$$

$$\Rightarrow F_x = -Gmx2\pi\sigma\int_0^a \frac{r\,dr}{(x^2+r^2)^{3/2}} = -2\pi Gmx\sigma\left[-(x^2+r^2)^{-\frac{1}{2}}\right]\Big|_0^a$$

$$= 2\pi Gmx\sigma\left(\frac{1}{\sqrt{x^2+a^2}} - \frac{1}{x}\right)$$, for $x>0$

for all x: $$F_x = 2\pi Gmx\sigma\left(\frac{1}{\sqrt{x^2+a^2}} - \frac{1}{|x|}\right)$$

, $(1+\varepsilon)^n \approx 1+n\varepsilon$