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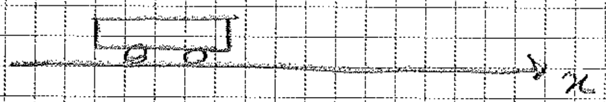
# Chapter 2: Motion along a Straight Line

We will consider a single particle moving along a straight line and analyze its motion.

We will use terms as $\begin{cases} \text{displacement} \\ \text{velocity} \\ \text{acceleration} \end{cases}$

## 2.1. Displacement, Time and Average Velocity:

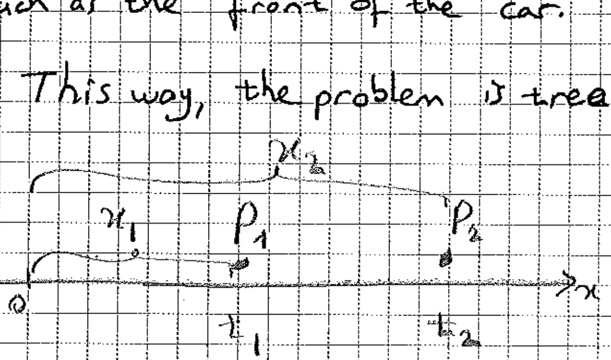
Consider a car moving along a straight track.




In order to analyze the motion of the car: - we choose the x-axis of the coordinate system along the straight track, we also choose an origin point.

- We represent the moving car by a representative point, such as the front of the car.

This way, the problem is treated as single particle motion on a straight line (in 1D).




Suppose the car is at point $P_1$ at time $t_1$ and at point $P_2$ at time $t_2$.

Displacement: The vector pointing from $P_1$ to $P_2$ with a magnitude equal to the change in the value of x.

$\vec{\Delta x}$: displacement $\quad \vec{\Delta x} = \vec{x_2} - \vec{x_1}$, in general.

For motion along a straight line (1D) the vector notation is not necessary:

$$\Delta x = x_2 - x_1,$$

If the particle is moving, we need to include the speed of the particle into our analysis.

Consider the car travels a displacement $\Delta x$ between the time instants $t_1$ and $t_2$.

We define the average velocity of the car as the displacement divided by the time interval.

$$\boxed{v_{av-x} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}}$$

Note: Average velocity is also a vectoral quantity. Since we analyze the 1D problem we neglect the vector notation, but refer to the average velocity as $v_{av-x}$ ← x-axis.

Examples:

- Consider a particle moving from $x_1 = 50m$ to $x_2 = 100m$ between time instants $t_1 = 1s$, $t_2 = 6$.

$$\Rightarrow \text{Average velocity } v_{av-x} = \frac{100m - 50m}{6s - 1s} = 10 \, m/s$$

- Consider a particle moving from $x_1 = 100m$ to $x_2 = 50m$, between time instants $t_1 = 1s$, $t_2 = 6s$

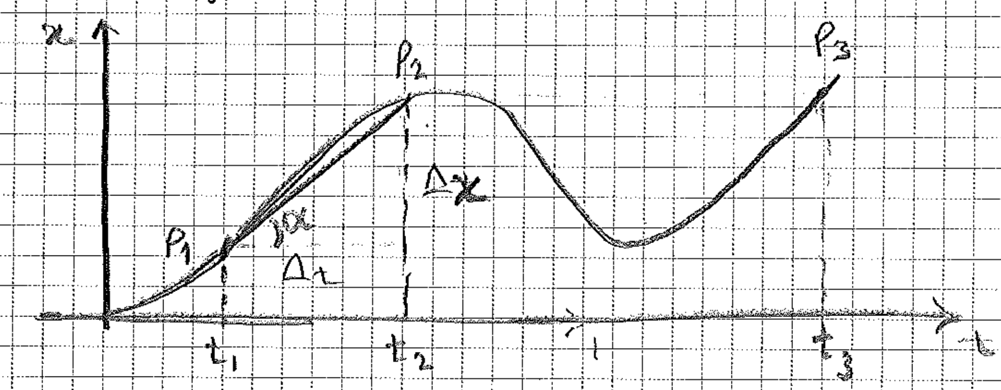
$$\Rightarrow \text{Average velocity}: v_{av-x} = \frac{50m - 100m}{6s - 1s} = -10 \, m/s$$

Hence: Average velocity can be a negative number.

Positive average velocity $\Rightarrow$ motion in $+x$ direction

Negative average velocity $\Rightarrow$ motion in $-x$ direction.

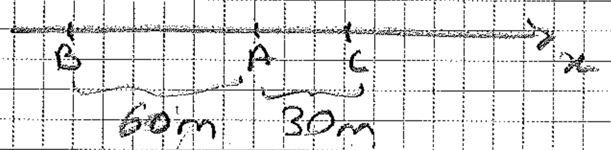
Consider a graph which shows the position of a particle as a function of time.




Average velocity between two points $P_1, P_2$ is the slope of the line ~~connecting~~ $P_1 P_2$.

$$\tan\alpha = v_{av-x} = \frac{\Delta x}{\Delta t}$$

Example:




Consider the points A, B, C shown above. What are the average velocity?

(i) A car driving from A to B in 2sec:

$$v_{av-x} = -\frac{60m}{2s} = -30 m/s$$

(ii) A car driving from A to C then to B in 5s:

$$v_{av-x} = -\frac{60m}{5s} = -12 m/s$$

(iii) A car driving from A to C then back to A in 5s:

$$v_{av-x} = \frac{0m}{5s} = 0 m/s$$

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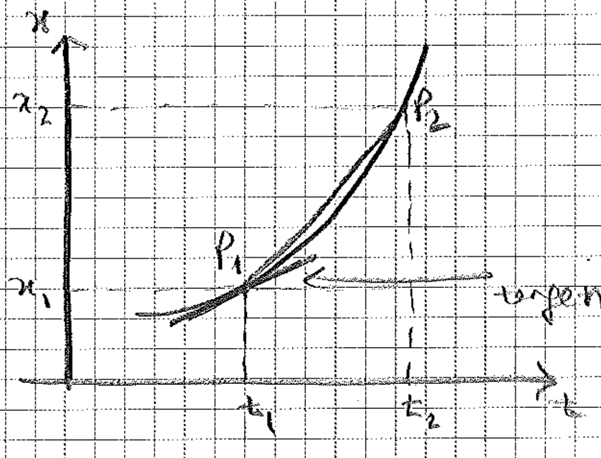
## 2.2. Instanteneous Velocity:

To describe the motion in greatest detail, we need to define the velocity at any specific instants of time or specific point along the path. Such a velocity is called as the instanteneous velocity.




$$v_{av\text{-}x} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$v_x(t_1) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

derivative of x with respect to t.

Instanteneous velocity is the limit of the average velocity as the time interval approaches zero,

In a graph of x vs. t, instanteneous velocity at $t = t_1$ is the slope of the line tangent to the graph at $t = t_1$.

Essentials on Derivatives:

In general $\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}$

ex: $- f(t) = t^2 \Rightarrow \Delta f = f(t+\Delta t) - f(t) = (t+\Delta t)^2 - t^2$

$$= \cancel{t^2} + 2(\Delta t)t + \Delta t^2 - \cancel{t^2} = 2t\Delta t + \Delta t^2$$

$$\Rightarrow \frac{\Delta f}{\Delta t} = \frac{2t\cancel{\Delta t} + \Delta t^2}{\cancel{\Delta t}} = 2t + \Delta t, \quad \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0}(2t + \cancel{\Delta t}^{0}) = 2t$$

$- f(t) = t^n \Rightarrow \frac{df}{dt} = nt^{n-1}$

$- f(t) = \sin t \Rightarrow \Delta f = \sin(t+\Delta t) - \sin t = \sin t \cos\Delta t + \cos t \sin\Delta t - \sin t$

$$\Rightarrow \frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\sin t \overbrace{\cos\Delta t}^{1} + \cos t \overbrace{\sin\Delta t}^{\Delta t} - \sin t}{\Delta t} = \lim_{\Delta t \to 0} \frac{\cos t \, \Delta t}{\Delta t} = \cos t$$

$f(t) = \cos t \Rightarrow \Delta f = \cos(t+\Delta t) - \cos t = \cos t \cos\Delta t - \sin t \sin\Delta t - \cos t$

$$\Rightarrow \frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{\cos t \overset{1}{\widehat{\cos\Delta t}} - \sin t \overset{\Delta t}{\widehat{\sin\Delta t}} - \cos t}{\Delta t} = -\sin t$$

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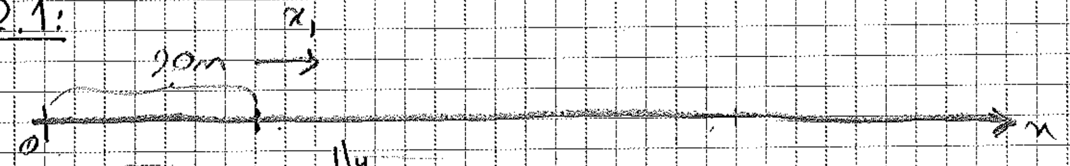
## Speed vs. Velocity:

**Speed**: is used to denote the distance travelled divided by time.
Speed gives an indication of how fast a particle is moving.

Velocity is used to describe how fast and in what direction a particle is moving.

ex: A particle with instanteneous velocity $v_x = 25\,m/s$ and another one moving in opposite direction with instanteneous velocity $v_x = -25\,m/s$ both have the same speed!

## Ex 2.1:




A particle is initially located at $x = 20\,m$. Position of the particle as a function of time is given as: $x(t) = 20\,m + 5\left(\frac{m}{s^2}\right)t^2$.

a) Find the displacement of the particle between $t_1 = 1s$, and $t_2 = 2s$.

$\Delta x = x_2 - x_1 = 20 + 5 \times 4 - (20 + 5 \times 1) = 15\,m$

b) Find the average velocity between $t_1$ and $t_2$:

$$v_{av-x} = \frac{\Delta x}{\Delta t} = \frac{15\,m}{2s - 1s} = 15\,m/s$$

c) Find the instanteneous velocity at $t_1 = 1\,s$ and $t_2 = 2s$

In general $v_x = \frac{dx}{dt} = 10\left(\frac{m}{s^2}\right)t$

$$\Rightarrow v_x(t_1) = 10\,\frac{m}{s}, \quad v_x(t_2) = 20\,\frac{m}{s}$$

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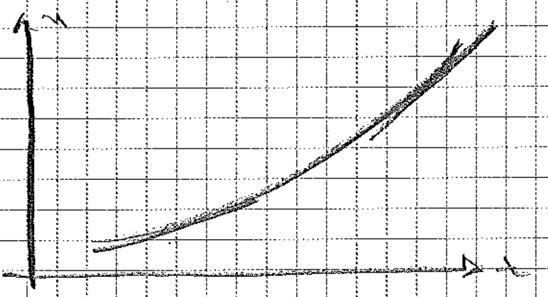
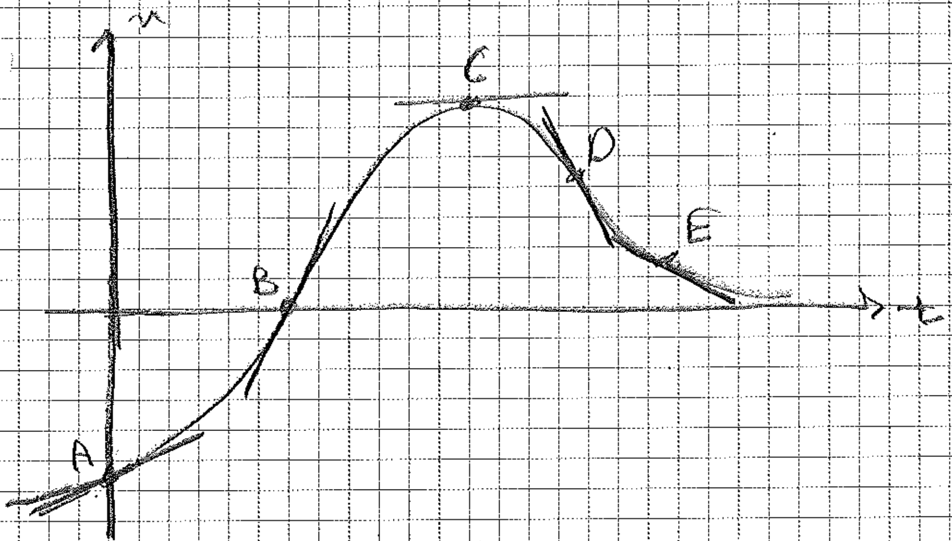
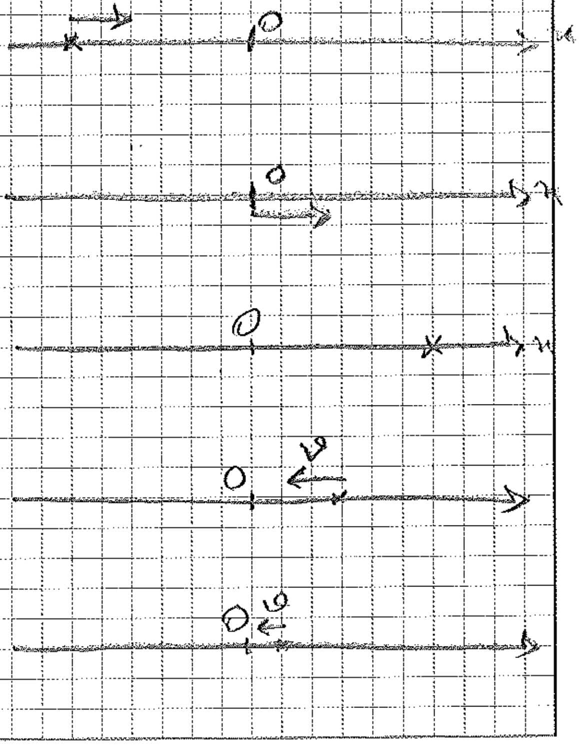
## Finding velocity on x-t Graph:




On a graph of position as a function of time, the instantaneous velocity at any point is equal to the slope of the tangent to the curve at that point.




| | motion of the particle |
|---|---|
| A | $v_x > 0$, moving in +x |
| B | $v_x > 0$, $x = 0$, moving in +x |
| C | $v_x = 0$, particle at rest |
| D | $v_x < 0$, moving in −x |
| E | $v_x < 0$, moving in −x |
| | $\lvert v_E \rvert < \lvert v_D \rvert$ |



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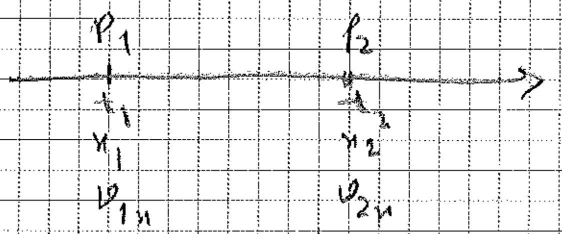
## 2.3. Average and Instanteneous Acceleration:

A particle has an acceleration when its velocity changes with time.

As velocity is the rate of change of position with time, acceleration is the rate of change of velocity with time.

### Average Acceleration:

Suppose at time $t_1$ a particle is at point $P_1$ with instanteneous velocity $v_{1x}$, and at time $t_2$ the particle moves to the point $P_2$ with instanteneous velocity $v_{2x}$:




The average acceleration of the particle as it moves from $P_1$ to $P_2$ is:

$$\boxed{a_{av\text{-}x} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t}} \Leftarrow$$

x component of a vector.

Units: velocity is expressed in $\frac{m}{s}$, acceleration in $\frac{m}{s^2}$.

### Instanteneous Acceleration:

Instanteneous acceleration is the limit of the average acceleration as the time interval approaches zero.

$$\boxed{a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}}$$

$\Rightarrow$ Instanteneous acceleration is also a vector.

Ex 2.3: Suppose the velocity of a car is given by the equation:

$$v_x(t) = 60\,\frac{m}{s} + 0.5\left(\frac{m}{s^3}\right)t^2$$

a) Find the change in velocity of the car in the time interval between $t_1 = 1s$, and $t_2 = 3s$

$$\Delta v_x = v_2 - v_1 = \cancel{60\,\frac{m}{s}} + 0.5\,\frac{m}{s^3}(4s)^2 - \cancel{60\,\frac{m}{s}} - 0.5\,\frac{m}{s^3}\,1s^2$$

$$= 1.5\,\frac{m}{s}$$

b) Find the average acceleration in this time interval:

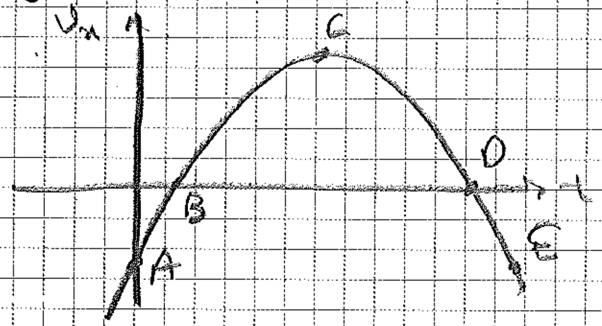
$$a_{av-x} = \frac{\Delta v_x}{\Delta t} = \frac{1.5\,m/s}{2s - 1s} = 1.5\,m/s^2$$

c) Find the instantaneous acceleration at $t = 1s$ and $t = 3s$.

$$a_x = \frac{dv_x}{dt} = t\left(\frac{m}{s^2}\right) \Rightarrow \text{at } t_1 = 1s,\ a_x(t_1) = 1\,m/s^2$$

$$t_2 = 3s,\ a_x(t_2) = 3\,m/s^2$$

Analyzing the $v_x - t$ Graph:




slope of the line tangent to the curve gives the acceleration

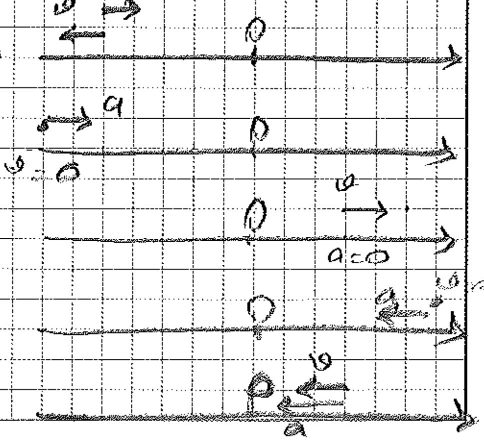
A $v_x < 0$, $a_x > 0$, moving in $-x$ direction slowing down

B $v_x = 0$, $a_x > 0$, at rest about to move in $+x$

C $v_x > 0$, $a_x = 0$, moving in $+x$ dir. moving at cons. speed

D $v_x = 0$, $a_x < 0$, at rest about to move in $-x$

E $v_x < 0$, $a_x < 0$, moving in $-x$ dir speeding up



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## Analyzing the $x$-$t$ Graph:

We can also learn about the acceleration of a body from a graph of its position versus time. Because $a_x = \frac{dv_x}{dt}$, $v_x = \frac{dx}{dt}$ we can write:

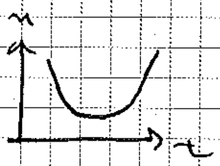
$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

Second derivative of $x$ with respect to $t$.

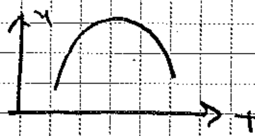
Second derivative is directly related to concavity or curvature of the graph of a function.

Properties:

- At a point where the $x$-$t$ graph is concave up (curved upward) the acceleration is positive ~~and $v_x$ is increasing~~.

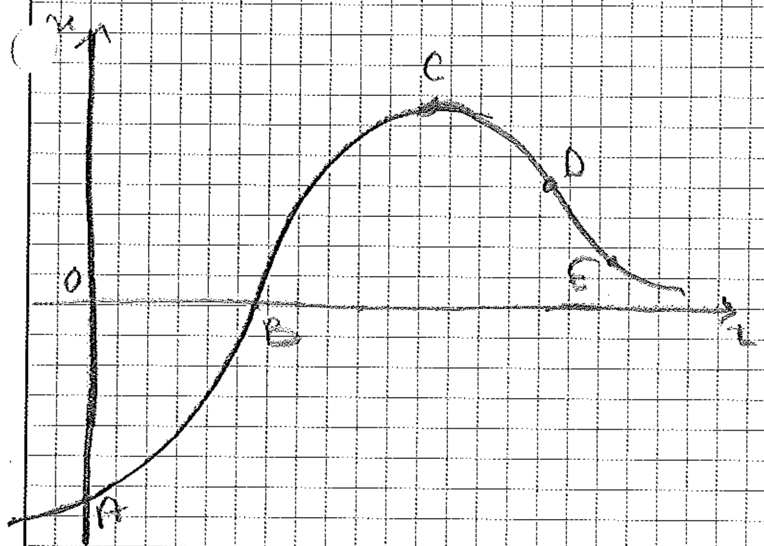


- At a point where the $x$-$t$ graph is concave down (curved downward), the acceleration is negative ~~and $v_x$ is decreasing~~.



- At a point where the $x$-$t$ graph has no curvature (line) such as an inflection point, the acceleration is 0.

Example:




| | $x$-$t$ graph | motion of the particle |
|---|---|---|
| A | positive slope upward curve. $v_x > 0$, $a_x > 0$ | moving in +x dir. speeding up |
| B | $v_x > 0$, $a_x = 0$ | moving in +x dir. cons. speed |
| C | $v_x = 0$, $a_x < 0$ | instantaneously at rest, about to move in −x |
| D | $v_x < 0$, $a_x = 0$ | moving in −x cons. speed |
| E | $v_x < 0$, $a_x > 0$ | moving in −x slowing down |

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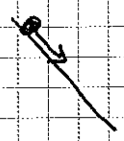
## 2.4. Motion with Constant Acceleration:

This is the simplest accelerated motion. But, it is observed quite often in nature.

Example: – A falling body (if the effects of the air are neglected, the motion of a falling body is motion with constant acceleration.

– A body sliding on an incline or along a rough horizontal surface also makes motion with constant acceleration.



We now derive the equations for position and velocity as functions of time.

in motion with constant acceleration: $a_{av-x} = a_x$

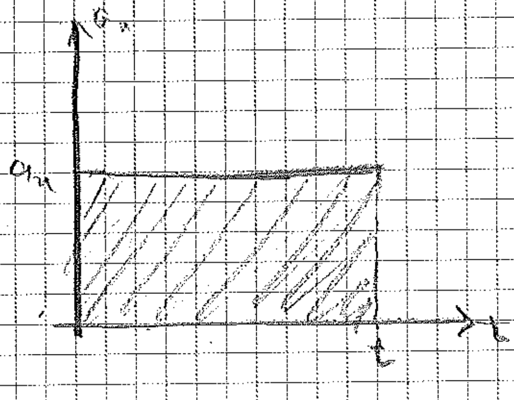
$$\Rightarrow \quad a_{av-x} = \frac{\Delta v_x}{\Delta t} = a_x = \frac{v_{2x} - v_{1x}}{t_2 - t_1}$$

Let $t_1 = 0$, and $t_2$ be any later time $t$.

If the particle's velocity at $t_1 = 0$ is $v_{0x}$, velocity at time $t$ is given by:

$$a_x = \frac{v_x - v_{0x}}{t - 0} \Rightarrow \boxed{v_x = v_{0x} + a_x t}$$

Graphical Interpretation:




The change in velocity of the particle $(v_x - v_{0x})$ between $t = 0$ and any later time $t$ equals the area under the $a_x$-$t$ graph between those times.

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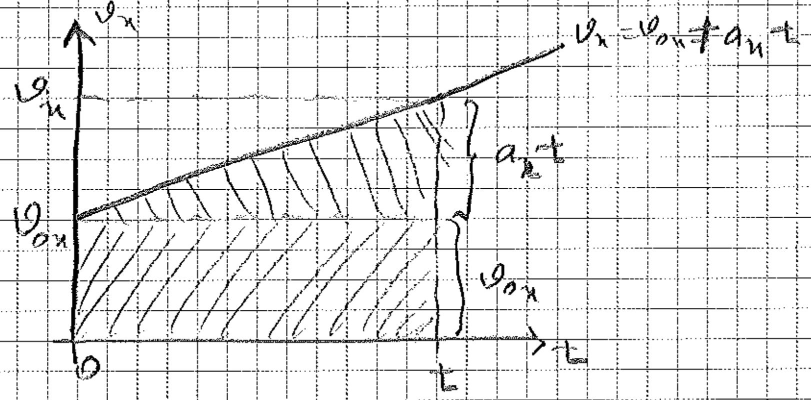
Next, we derive an equation for the position $x$ of a particle moving with constant acceleration.




At any time t:

$$v_x = \frac{dx}{dt} \Rightarrow \underbrace{\int_0^t v_x\, dt}_{\text{Area under the } v_x - t \text{ graph between } 0 \text{ and } t.} = x(t) - x(0)$$

$$\int_0^t v_x\, dt = v_{0x} t + \frac{a_x t^2}{2} = x(t) - x_0 \Rightarrow \boxed{x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2}$$

check: $v_x = v_{0x} + a_x t \Rightarrow a_x = \frac{dv_x}{dt} = a_x$ ✓

$x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \Rightarrow v_x(t) = \frac{dx}{dt} = v_{0x} + a_x t$ ✓

Using the equation, $v_x = v_{0x} + a_x t$ and $x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$ we can derive a relationship between $x, v_x, a_x$ that does not involve time.

$$v_x = v_{0x} + a_x t \Rightarrow t = \frac{v_x - v_{0x}}{a_x}$$

$$\Rightarrow x = x_0 + v_{0x}\left(\frac{v_x - v_{0x}}{a_x}\right) + \frac{1}{2} a_x \left(\frac{v_x - v_{0x}}{a_x}\right)^2$$

$$\Rightarrow 2a_x x = 2a_x x_0 + 2v_{0x}(v_x - v_{0x}) + v_x^2 + v_{0x}^2 - 2v_x v_{0x}$$

$$\Rightarrow 2a_x(x - x_0) = v_x^2 + v_{0x}^2 - 2v_{0x}^2 = v_x^2 - v_{0x}^2 \Rightarrow \boxed{v_x^2 = v_{0x}^2 + 2a_x(x - x_0)}$$

$\boxed{v_x^2 = v_{0x}^2 + 2a_x(x - x_0)}$ is valid for motion with constant acceleration only.

Another useful relationship for motion with constant acceleration is:

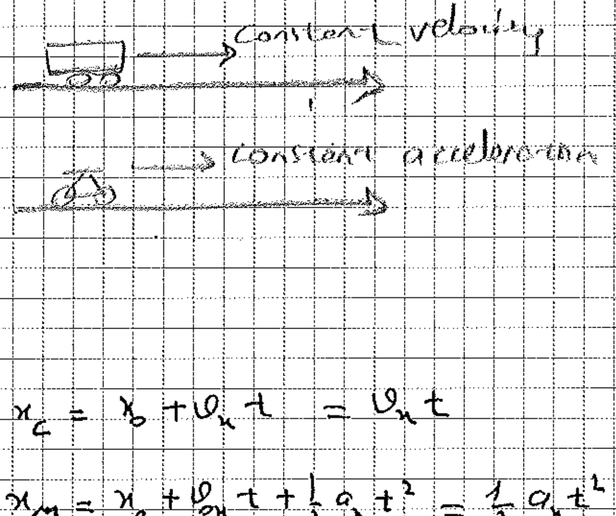
$$x(t) = x_0 + \left(v_{0x} + \frac{1}{2}a_x t\right)t$$

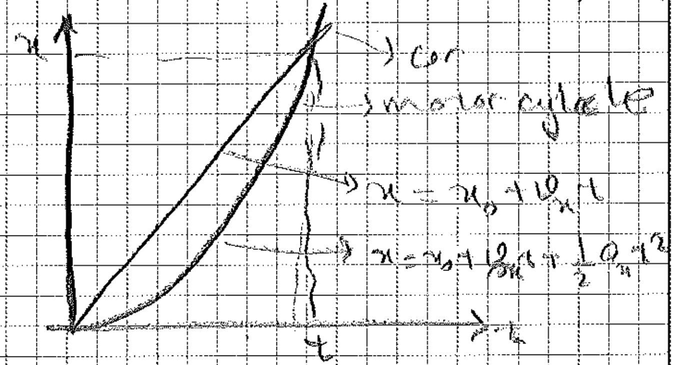
$$= x_0 + \left(\cancel{v_{0x}} + \frac{v_{0x} + a_x t}{2}\right)t = \boxed{x_0 + \left(\frac{v_{0x} + v_x}{2}\right)t}$$

## Example 2.5.

A car has a constant velocity of 15 m/s. Just as the car passes, a motorcycle starts to chase the car with a constant acceleration of 3 m/s².

(a) How much time elapses before the motorcycle catches up with the car?
(b) What is the motorcycle's speed at that time?
(c) What is the total distance each vehicle has travelled at that point?




$$\left.\begin{aligned} x_c &= x_0 + v_x t = v_x t \\ x_m &= x_0 + v_{0x}t + \frac{1}{2}a_x t^2 = \frac{1}{2}a_x t^2 \end{aligned}\right\}$$

At time t when they meet:

$$v_x t = \frac{1}{2}a_x t^2$$

$\Rightarrow t = 0$ (trivial) or $t = \dfrac{2v_x}{a_x} = \dfrac{2 \times 15\,m/s}{3\,m/s^2} = \boxed{10\,s}$

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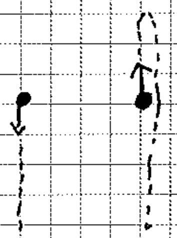
b) $v_M = \underbrace{v_{0x}}_{0} + a_x t = 3\,m/s^2 \times 10\,s = 30\,m/s$ //

c) $x_M = x_C = v_C \times t = 15\,m/s \times 10\,s = 150\,m$ //

## 2.5 Freely Falling Bodies:



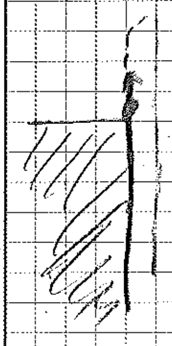
When we neglect the effects of air, ~~the Earth's rotation and the decrease of acceleration with increasing altitude~~, motion of freely falling bodies can be approximated as motion with constant acceleration.

- The constant acceleration of a free falling body is called the acceleration due to gravity:

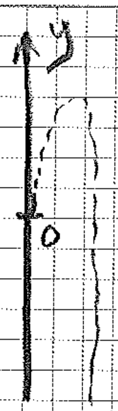
$$g = 9.8\,m/s^2$$ ← The exact value of $g$ varies with latitude, but we will use this value as an approximation.

- The direction of $g$ is towards the earth.

**Example 2.7:** You throw a ball vertically upward from the roof of a tall building with an upward speed of $15\,m/s$. Find
a) The position and velocity of the ball 1 s and 4 s after it is thrown
b) The velocity when the ball is 5 ~~m~~ above
c) The maximum height reached and the time at which it is reached
d) Acceleration of the ball when it is at max. height.



In analyzing the motion free falling bodies always choose the vertical axis as the axis of motion.




Then the motion is governed by the equations:

$$\boxed{\begin{aligned} v_y &= v_{0y} - g t \\ y &= y_0 + v_{0y} t - \frac{1}{2} g t^2 \end{aligned}}$$

Let's choose $y_0 = 0$, and $v_{0y} = 15\,m/s$ is given.

a) $v_y = v_{0y} - g t \Rightarrow v_y(1) \approx 15\,m/s - 9.8\,m/s = 5.2\,m/s$

$v_y(4) = 15\,m/s - 9.8\,m/s^2 \times 4s \approx -25\,m/s$

$y(1) \approx 0 + 15\,m/s \times 1s - \frac{1}{2} 10 \approx 10\,m$ //

$y(4) = 15\,m/s \times 4s - \frac{1}{2} 10 \times 16 = 60\,m - 80\,m = -20\,m$ //

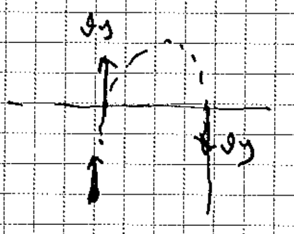
b) $y = 5m = 15\,m/s\, t - \frac{1}{2} \times 10 \times t^2 \Rightarrow 5t^2 - 15t + 5 = 0$

$\Rightarrow t = \dfrac{15 \pm \sqrt{225 - 100}}{10} = \dfrac{15 \pm 11.18}{10} = 2.6\,s$

or $0.4\,s$

$\Rightarrow v_y = v_{0y} - g t = 15\,m/s - 10 \times 2.6 = -11\,m/s$

or $v_y = 15\,m/s - 10 \times 0.4 = 11\,m/s$




The ball passes this point twice.

alternatively use: $v_y^2 = v_{0y}^2 + 2a_y(y - y_0) = v_{0y}^2 - 2g(y - y_0)$

$= (15)^2 - 2 \times 10 \times (5 - 0) = 225 - 100 = 125$

$\Rightarrow \boxed{v_y = \pm 11\,m/s}$

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22.5

c) At the maximum point $v_y = 0$.

$\Rightarrow v_y = v_{0y} - gt = 0 \Rightarrow t = \frac{v_{0y}}{g} = \frac{15\ m/s}{10\ m/s^2} = 1.5\ s$ //

$y = v_{0y}t - \frac{1}{2}gt^2 = 15 \times 1.5 - \frac{1}{2} 10 \times (1.5)^2 = 22.5 - 5 \times 2.25 = \boxed{11.25}\ m$

d) $-g$, always!