Chapter 7: Potential Energy and Energy Conservation

The work done by some forces only depends on the initial and final position of a particle. Such forces are called conservative forces.

Examples of conservative forces are the gravitational force or spring force.

When the work done by a force depends on the path followed by the particle such a force is called a nonconservative force.

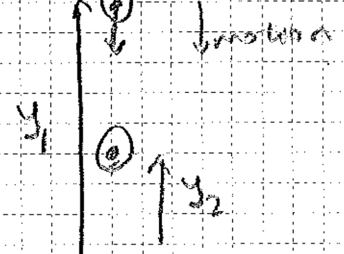
Example of a nonconservative force is the friction force.

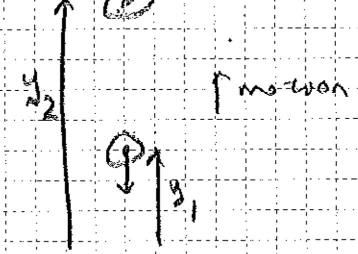
For conservative forces, it is possible to define a potential function $U(\vec{r})$ such that the work done by the force is given as:

$$W_F = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{l} = U(\vec{r}_{P_1}) - U(\vec{r}_{P_2})$$

7.1. Gravitational Potential Energy:

Let's calculate the work done by the gravitational force.





$W_{grav} = mg(y_1 - y_2)$ $\qquad$ $W_{grav} = -mg(y_2 - y_1) = mg(y_1 - y_2)$

$$\Rightarrow \boxed{W_{grav} = mgy_1 - mgy_2}$$

"$U = mgy$" is called the gravitational potential energy.

$$\Rightarrow W_{grav} = U_1 - U_2 = -(U_2 - U_1) = -\Delta U$$ ← Work done by the gravitational force.

- When the body moves up $y_2 > y_1$ ⇒ gravitational potential energy increases ($\Delta U > 0$)
- When the body moves down $y_2 < y_1$ ⇒ gravitational potential energy decreases ($\Delta U < 0$)

If the only force applied to a body is gravitational force:

work-energy theorem $W_{grav} = \Delta K = -\Delta U \Rightarrow K_2 - K_1 = U_1 - U_2$

$$\Rightarrow \boxed{K_1 + U_1 = K_2 + U_2}$$

define $E = K + U$: Total mechanical energy of the system.

$\Rightarrow$ $E = K + U$ is constant: When only the force of gravity does work, the total mechanical energy is conserved.

Example 7.1: You throw a 0.145 kg ball straight up, with an upward initial velocity of 20 m/sec.
How high does the ball go? (ignore air resistance)

Conservation of energy:

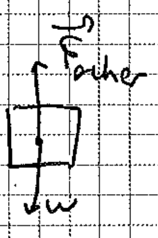
$$E_1 = K_1 + U_1 = \frac{1}{2} m v_0^2 + mgy_1 = K_2 + U_2 = \frac{1}{2} m 0^2 + mgy_2$$

$$\Rightarrow \frac{1}{2} \cancel{m} v_0^2 = \cancel{m} g(y_2 - y_1) \Rightarrow \boxed{y_2 - y_1 = \frac{v_0^2}{2g}}$$

y_1 can be arbitrarily chosen

$\Rightarrow y_2 - y_1 = 20.4\,m$

Effect of Other Forces



What happens when other forces ($\vec{F}_{other}$) act on the body in addition to the gravitational force?

$\Rightarrow$ Write the work-energy theorem (is always valid)

$$W_{grav} + W_{other} = \Delta K$$

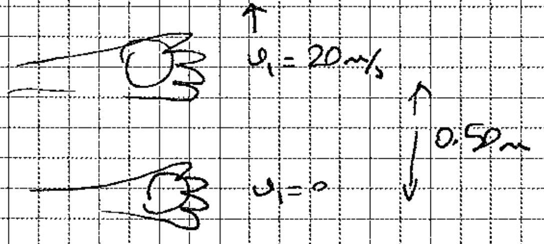
$\Rightarrow \quad W_{grav} = U_1 - U_2$

$\Rightarrow \quad K_1 + U_1 + W_{other} = K_2 + U_2$

$$\boxed{\frac{1}{2} m v_1^2 + mgy_1 + W_{other} = \frac{1}{2} m v_2^2 + mg y_2} \Rightarrow \boxed{W_{other} = E_2 - E_1}$$

"The work done by all forces other than the gravitational force is equal to the change in the total mechanical energy of the system."

Example 7.2: (Maybe skip)



Suppose your hand moves up 0.5 m while exerting a constant force to the ball.

a) Magnitude of the force applied by your hand?

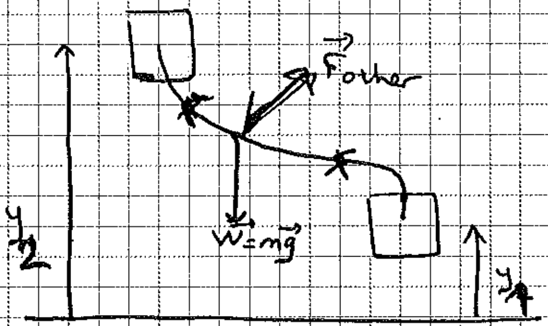
b) Speed of the ball 15 m above the point where it leaves your hand?

(a) $\quad W + \frac{1}{2} \cancel{m v_1^2}_{0} = \frac{1}{2} m v_2^2 + mg(y_2 - y_1) = F(y_2 - y_1)$

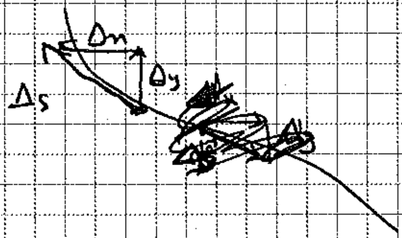
(b) $\quad \frac{1}{2} m v_2^2 + mgy_2 = \frac{1}{2} m v_3^2 + mgy_3 \Rightarrow \frac{1}{2} v_3^2 = \frac{1}{2} v_2^2 + g(y_2 - y_3)$

$$= \frac{1}{2} v_2^2 - g(y_3 - y_2)$$

Gravitational Energy for Motion Along a Curved Path:



$$W_{grav} = \int_{P_1}^{P_2} \vec{w}\cdot d\vec{l}$$



$$\Delta\vec{s} = \Delta x\,\hat{x} + \Delta y\,\hat{y}$$

Consider an infinitesimal displacement $\Delta\vec{s}$

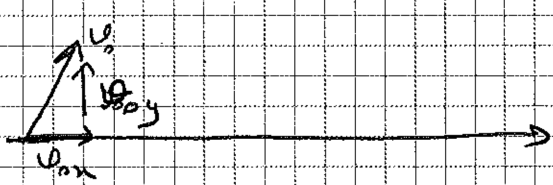
$$\Delta W_{grav} = \vec{w}\cdot\Delta\vec{s} = -w\Delta y$$

$$\Rightarrow W_{grav} = -\int_{P_1}^{P_2} w\,dy = -w(y_2-y_1) = \boxed{U_1-U_2}$$

Work done by the gravitational force is unaffected by horizontal motion.

Ex. 7.4: Maximum height of a projectile:

At maximum height: $v = v_{0x}$

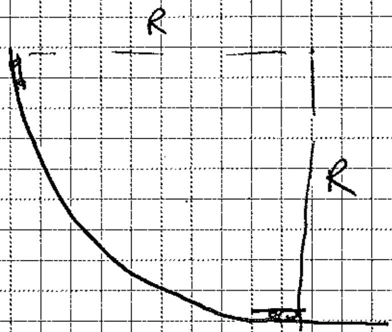


$$K_1+U_1 = K_2+U_2$$

$$\Rightarrow \frac{1}{2}mv^2 + mg0 = \frac{1}{2}mv_x^2 + mgh$$

$$\Rightarrow mgh = \frac{1}{2}m(v^2 - v^2\cos^2\alpha) \Rightarrow h = \frac{v^2}{2g}(1-\cos^2\alpha) = \frac{v^2}{2g}\sin\alpha$$

Ex. 7.5: - Fun Swing -



$m = 25\,kg$.

a) Find the speed at the bottom of the ramp

b) Normal force that acts on him at the bottom of the curve.

a) $K_1 = 0 \quad U_1 = mgR$
$K_2 = \frac{1}{2} m v_2^2 \quad U_2 = 0$ $\Rightarrow K_1 + U_1 = K_2 + U_2$

normal force does no work, because it is always perpendicular to the motion.

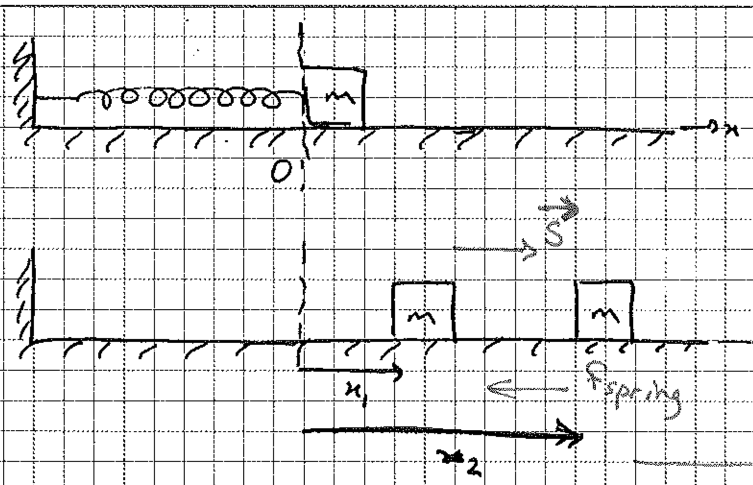
b) At the bottom of the curve
non-uniform circular motion;

$$a_{rad} = \frac{v^2}{R} = \frac{n - w}{m} \Rightarrow \boxed{n = w + m\frac{v^2}{R}}$$

7-2) Elastic Potential Energy

Springs can also store energy which can later be converted to kinetic energy. This will be described by elastic potential energy.

We will assume an ideal spring: $F = kx$ needed to stretch the spring to "x" from the unstretched position



Imagine you pull the mass m in +x direction.

Work done by the spring: $\boxed{W_{el} = \int_{x_1}^{x_2} -kx\,dx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2}$

When $|x_1| < |x_2|$ spring does negative work on the block

$|x_1| > |x_2|$ ——— positive work on the block.

Define the elastic potential energy:

$$U = \frac{1}{2}kx^2$$

$$\Rightarrow \boxed{W_{el} = U_1 - U_2 = -\Delta U}$$

When a spring is stretched $W_{el} > 0 \rightarrow \Delta U > 0$: stored energy

——— compressed $W_{el} < 0 \rightarrow \Delta U < 0$: spring loses its potential energy.

$\Rightarrow$ According to work-energy theorem

$$W_{total} = \Delta K$$

if the elastic force is the only force that does work on a body,

$$W_{total} = W_{el} = U_1 - U_2 = K_2 - K_1 \Rightarrow U_1 + K_1 = U_2 + K_2$$

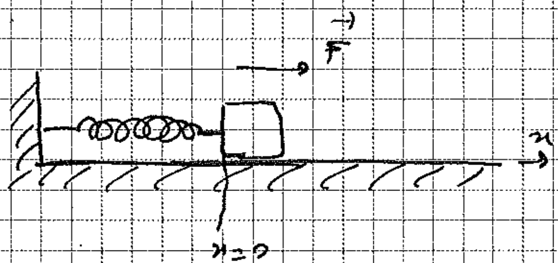
$E = K + U$: total mechanical energy is conserved.

If forces other than the elastic force do work:

$$W_{tot} = W_{el} + W_{other} = \Delta K$$

$$\Rightarrow \boxed{K_1 + U_1 + W_{other} = K_2 + U_2}$$

Ex 7.9:



Suppose the mass is initially at rest at x=0.
You apply a constant $\vec{F}$ in +x direction with magnitude 0.610 N

What is the mass's velocity at x=0.1m?

$$W_{others} + W_{spring} = \Delta K$$

$$W_{others} + \underset{0}{U_1} + \underset{0}{K_1} = K_2 + U_2 \Rightarrow W_{others} = (0.610N)(0.1m) = K_2 + \frac{1}{2}kx_2^2$$

$$\Rightarrow K_2 = (0.610N)(0.1m) - \frac{1}{2}kx_2^2$$

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7.3. Conservative and Nonconservative Forces:

A force which allows for two-way conversion between kinetic and potential energies is called a conservative force.

When a conservative force is applied on a body:

"Total mechanical energy" $E = K + U$ is conserved. (e.g. gravitational force, spring force)

"Law of conservation of energy"

Work done by a conservative force always has the property:

(i) $W_{cons} = U_1 - U_2$: It can be expressed as the difference between initial and final values of a potential energy function.

⇒ When starting and ending points are the same, total work done by a conservative force is zero.

Consider a conservative force in 1D:

$$W_{cons} = \int_{x_1}^{x_2} F(x)\,dx = U(x_1) - U(x_2)$$

if $G(x)$ is a function such that $\frac{dG}{dx} = F$

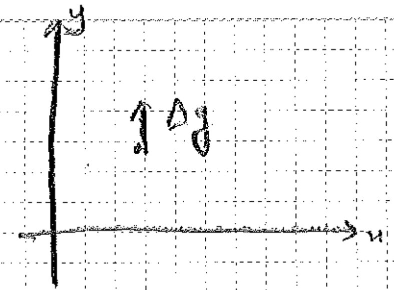
$$\Rightarrow \int_{x_1}^{x_2} F(x)\,dx = G(x_2) - G(x_1) = U(x_1) - U(x_2)$$
$$= -(U(x_2) - U(x_1))$$

$$\Rightarrow \boxed{F = -\frac{dU(x)}{dx}}$$

Relationship between force and potential energy function in 1D.

In general when $\vec{F}(x,y,z)$ is a conservative force in 3D:

$$\vec{F}(x,y,z) = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$



Work done to move the particle for a displacement of Δy along +y direction is:

$$F_y(x,y,z)\,\Delta y = U(x,y,z) - U(x,y+\Delta y,z)$$

$$\Rightarrow F_y = \frac{U(x,y,z) - U(x,y+\Delta y,z)}{\Delta y}$$

$$\Rightarrow F_y = \lim_{\Delta y \to 0} -\left(\frac{U(x,y+\Delta y,z) - U(x,y,z)}{\Delta y}\right) = \frac{\partial U}{\partial y}$$ ⟵ partial derivative of U with respect to y.

Therefore:

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = -\underbrace{\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)}_{\vec{\nabla} U \text{ : gradient of } U.} = -\vec{\nabla} U$$

$$\Rightarrow \boxed{\vec{F} = -\vec{\nabla} U}$$ relationship between the potential function and a conservative force

Examples:

Elastic potential energy: $U = \frac{1}{2}kx^2 \Rightarrow F_x = -\frac{dU}{dx} = -kx$ ✓

Gravitational potential energy: $U = mgy \Rightarrow F_y = -\frac{dU}{dy} = -mg$ ✓

A spring in 2D: $U = \frac{1}{2}k(x^2+y^2)$

$$\Rightarrow \vec{F} = -\vec{\nabla} U = -kx\hat{i} - ky\hat{j}$$

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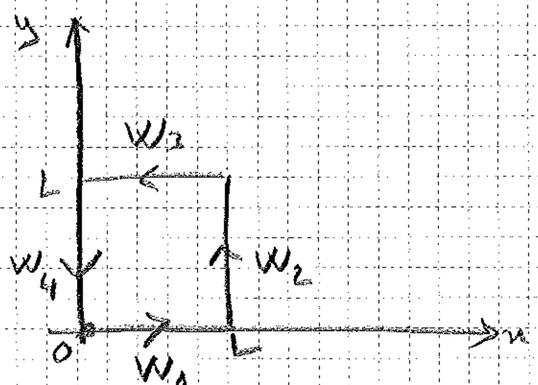
Ex 7.13: In a certain region of space the force on an electron is $\vec{F} = Cx\hat{j}$ where C is a constant. Is this force conservative?

Two ways of solving the problem:

(i) Find the work done by $\vec{F}$ along a closed loop, it should be equal to 0 if $\vec{F}$ is conservative.

(ii) Find a potential function U such that $\vec{F} = -\vec{\nabla}U$.

(i) Work done along a closed loop.



$W = \int \vec{F} \cdot d\vec{l}$ along the square in counterclockwise direction.

$d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$W_1 = \int (Cx\hat{j}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) = \int_{x=0}^{x=L} Cx\,dy = 0$$, no change in y.

$$W_2 = \int_{y=0}^{L} Cx\,dy = \int_0^L CL\,dy = CL\int_0^L dy = CL^2$$

$$W_3 = \int_{x=L}^{0} Cx\,dy = 0$$, no change in y.

$$W_4 = \int_{y=L}^{0} C\underset{\substack{\uparrow \\ 0}}{x}\,dy = 0$$, $(x=0)$

$\Rightarrow W = W_1 + W_2 + W_3 + W_4 = CL^2 \neq 0$

$\Rightarrow \vec{F}$ is non conservative.

(ii) $\vec{F} = Cx\hat{j} = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)$

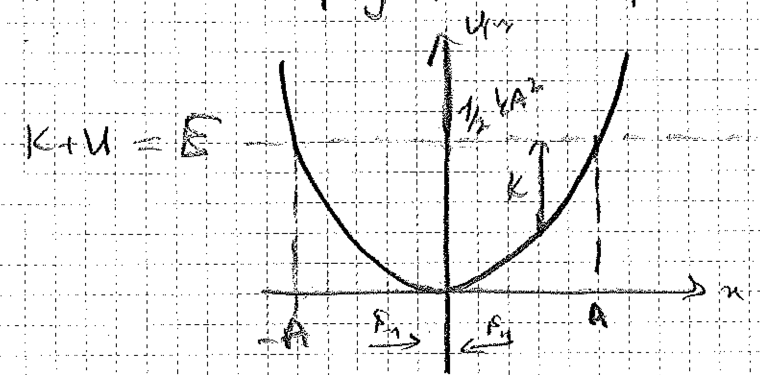
$\Rightarrow \frac{\partial U}{\partial y} = -Cx \Rightarrow U = -Cxy + U_0$

But if $U = -Cxy + U_0 \Rightarrow -\frac{\partial U}{\partial x} = Cy \neq F_y$, so a potential function U cannot be found.

7.5. Energy Diagrams:

When a particle moves in 1D under the influence of a conservative force, a lot of insight can be gained by analyzing the graph of the potential energy function $U(x)$.

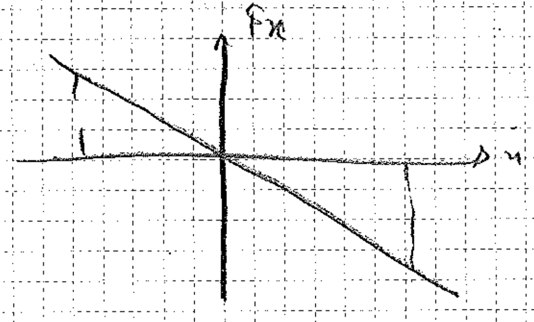
Consider the spring potential function $U(x) = \frac{1}{2}kx^2$



If the total energy of the particle is $E = \frac{1}{2}kA^2$

$\Rightarrow$ The particle moves between $x = -A$ and $x = A$

$F_x = -\frac{dU}{dx} = -kx$



F_x is trying to bring the particle towards $x=0$.

$\Rightarrow x=0$ is a point of stable equilibrium. Any minimum in a potential energy curve is a stable equilibrium position.

* The particle will oscillate around the stable equilibrium point.

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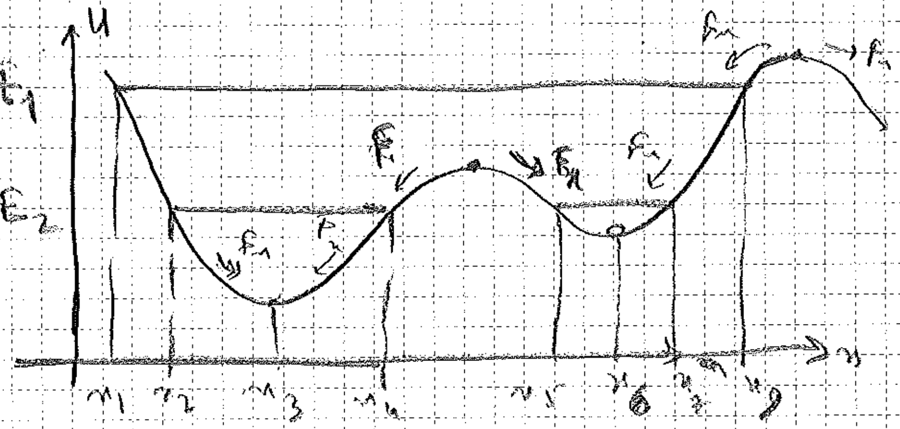
Now consider a general $U(x)$:

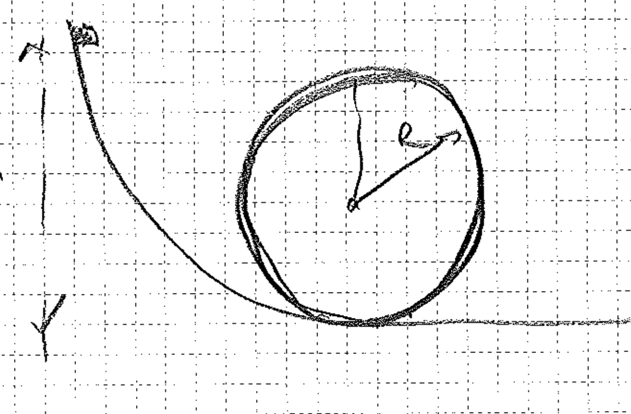


When the particle is located on a local maximum, $\frac{dU}{dx} = 0$, there is no force applied to the particle. A local maximum represents an equilibrium point where the particle may stay forever.
However F_x is trying to move the particle away from this equilibrium point $\Rightarrow$ A local maximum is an unstable equilibrium point.

If total energy is E_1
particle oscillates between x_1 and x_8

$E_2 \Rightarrow$ particle oscillates either between $x_2 - x_4$ or $x_5 - x_7$

Prob 7.46:



A car rolls without friction around the track shown.
Car is initially at rest at height h
a) What is the minimum value of h such that the car moves around the loop without falling?

$$\frac{mv^2}{R} = mg \Rightarrow v = \sqrt{Rg}$$

$$mgh = mg2R + \frac{1}{2} mRg$$

$$\Rightarrow h = 2R + \frac{1}{2}R = \frac{5}{2}R$$

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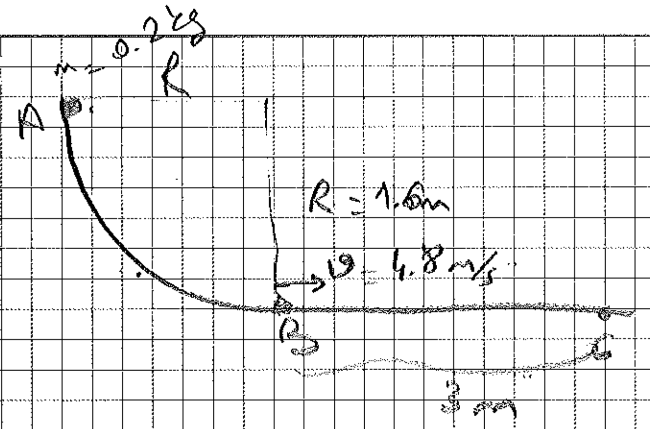
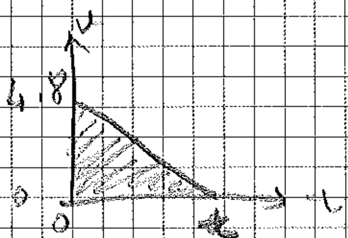


a) If the block stops at C, what is μ_k?



$$\frac{4.8t}{2} = 3m \Rightarrow t = \frac{3}{2.4} \text{ sec}$$

$$a = \frac{4.8}{\frac{3}{2.4}} = \frac{(4.8)(2.4)}{3}$$

$$\Rightarrow f = \cancel{0.2} \times \frac{(4.8)(2.4)}{3} = \mu_k \cancel{(0.2)} \times 9.8$$

$$\mu_k = \frac{(4.8)(2.4)}{3 \cdot (9.8)} = \ldots$$

b) Work done by friction on block as it slides from A to C?

$$W_f + W_{mg} = \Delta K = \frac{1}{2} m (4.8)^2$$

$$W_f = \frac{1}{2} \times 0.2 (4.8)^2 - \left((0.2)(9.8)(1.6) \right) = \ldots$$

$$W_{f_{BC}} = -\mu_k (0.2)(9.8) \times 3m = \ldots$$