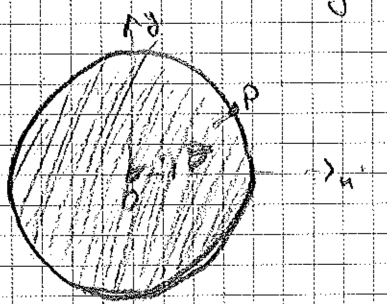
Chapter 9: Rotation of Rigid Bodies

In this chapter we will analyze the rotational motion of a rigid body.
A rigid body is a body which has a definite and unchanging shape and size.

9.1. Angular Velocity and Acceleration:

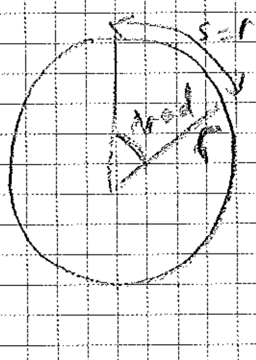
Consider a rigid body that rotates about a fixed axis.



The rotation of this body can be described by analyzing the motion of a particular point P on the rigid body.

We define the angular coordinate of P, as the angle θ that the line OP makes with the $+x$ axis. O is the point through which the axis of rotation passes.

We will measure the angle θ in "radians" instead of degrees.



1 radian is the angle at the center of a circle showing an arc of length "r".

$$\theta = \frac{s}{r}$$

$$s = 2\pi r \Rightarrow \theta = \frac{2\pi r}{r} = \boxed{2\pi \text{ rad} = 360^\circ}$$

Angular Velocity:

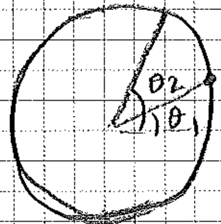
Angular velocity, $\vec{\omega}$, is a vector which describes the rate of change of θ.

The magnitude of $\vec{\omega}$ is defined as:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$: instantaneous angular velocity

units: $\left(\frac{rad}{sec}\right)$

$$\omega_{av} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$: average angular velocity



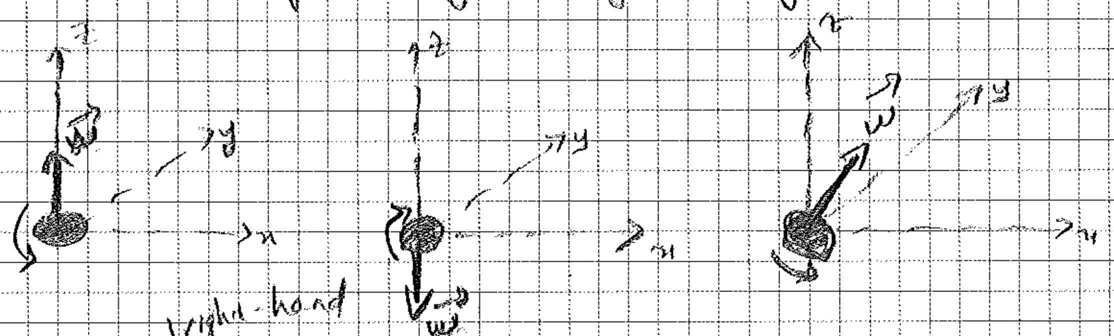
* It is important to note that when a body is rigid, all points on the body rotate through the same angle in the same time.

* At any instant, all points on a rotating body have the same angular velocity. ⇒ Angular velocity describes the rotation of the rigid body.

* Linear velocity is not the same at different points on a rotating rigid body.

Direction of $\vec{\omega}$:

Direction of $\vec{\omega}$ is given by the right-hand rule



When the right-hand fingers are curled in direction of rotation, the thumb points in direction of $\vec{\omega}$.

Angular Acceleration:

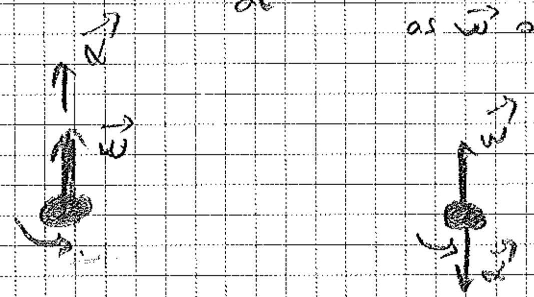
Angular acceleration, $\vec{\alpha}$, is the vector describing the rate of change of angular velocity, $\vec{\omega}$.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Magnitude of $\vec{\alpha}$: $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$ instantaneous angular velocity

$\alpha_{av} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$; average angular velocity.

Direction of $\vec{\alpha}$: $\vec{\alpha} = \frac{d\vec{\omega}}{dt} \Rightarrow \vec{\alpha}$ points either in the same direction as $\vec{\omega}$ or in the opposite direction.



Speeding up Slowing down.

Example: The angular position of a wheel is given by:

$$\theta = 2\left(\text{rad}/s^3\right)t^3$$

a) What is the instantaneous angular velocity at $t = 5s$?

b) Average angular acceleration between $t_1 = 2s$ and $t_2 = 5s$?

c) Instantaneous angular acceleration at $t = 5s$?

a) $\omega_z = 6t^2 \Rightarrow$ at $t = 5s$: $\omega_z(5s) = 150\ \text{rad}/s$

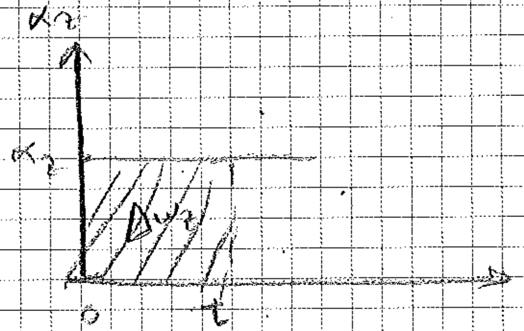
b) $\omega_z(2s) = 24\ \text{rad}/s$, $\omega_z(5s) = 150\ \text{rad}/s \Rightarrow \alpha_{av,z} = \frac{150 - 24}{5 - 2} = \frac{126}{3} = 42\ \text{rad}/s^2$

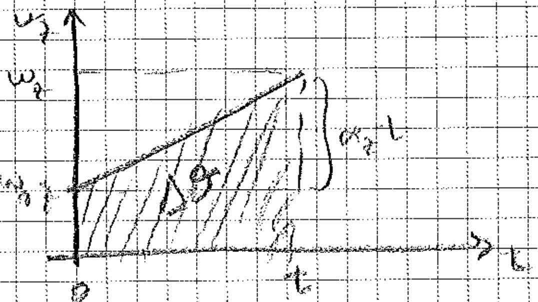
c) $\alpha_z = 12t \Rightarrow \alpha_z(5s) = 60\ \text{rad}/s^2$

9.2. Rotation with Constant Angular Acceleration:

The analysis is similar to the analysis of the straight-line motion with constant acceleration.

For $\alpha_z = \text{constant} = \dfrac{d\omega_z}{dt}$





$$\boxed{\omega_z(t) = \omega_{0z} + \alpha_z t} = \frac{d\theta}{dt}$$

$$\boxed{\theta(t) = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2}$$

similarly: $\theta = \theta_0 + \omega_{0z}\left(\dfrac{\omega_z - \omega_{0z}}{\alpha_z}\right) + \dfrac{1}{2}\alpha_z\left(\dfrac{\omega_z - \omega_{0z}}{\alpha_z}\right)^2$

$$\Rightarrow 2\alpha_z(\theta - \theta_0) = 2\omega_{0z}(\omega_z - \omega_{0z}) + (\omega_z - \omega_{0z})^2$$

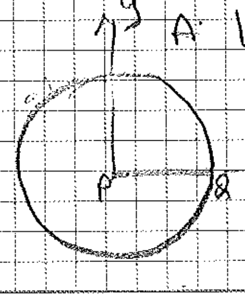
$$= (\omega_z - \omega_{0z})(\omega_z + \omega_{0z})$$

$$= \omega_z^2 - \omega_{0z}^2 \quad \Rightarrow \quad \boxed{\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)}$$

Ex. 9.3: A disc slows down with an angular acceleration $-10\ rad/s^2$.

At $t=0$ the angular velocity is $27.5\ rad/s$.

A line PQ on the surface of the disc lies along the x axis at $t=0$.



a) what is the disc's angular velocity at $t = 0.3s$

b) What angle does PQ make with the x-axis at $t = 0.3s$?

KOÇ ÜNİVERSİTESİ

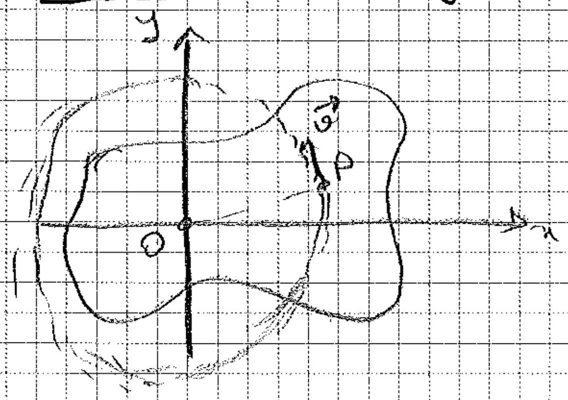
a) $\omega_z(t) = \omega_{0z} + \alpha_z t = 27.5 - 10t$

$$= 27.5 - 10 \times 0.3 = 24.5 \text{ rad/s}$$

b) $\theta(t) = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2$

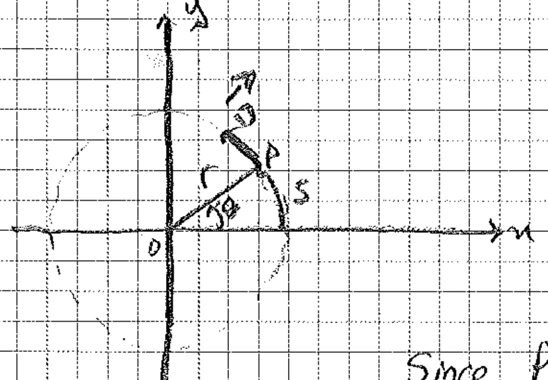
$$= 0 + (27.5)(0.3) - \frac{1}{2}10 \times (0.3)^2 = 7.8 \text{ rad}$$

9.3. Linear and Angular Kinematics:



Consider a rigid body rotating about a fixed axis.

We will express the linear velocity and acceleration at a point P in terms of the angular velocity and acceleration.



Let r be the distance from point P to the axis of rotation;

$\Rightarrow$ The arc spanned by P during an angular displacement θ is:

$$s = r\theta$$

Since P performs circular motion (non uniform) the velocity of P is along the tangential direction.

$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$: position vector

$$\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \left(-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}\right)\frac{d\theta}{dt} = r\omega_z\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$$

$\Rightarrow \boxed{v = r\omega_z}$, magnitude of v.

The direction of the linear velocity vector is tangential to the circular path at each point.

What about the acceleration?

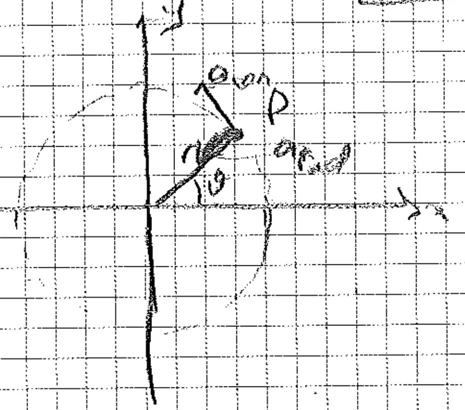
$$\vec{v} = r\omega(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$\Rightarrow \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left\{ r\omega(t)\left(-\sin(\theta(t))\,\hat{i} + \cos(\theta(t))\,\hat{j}\right)\right\}$$

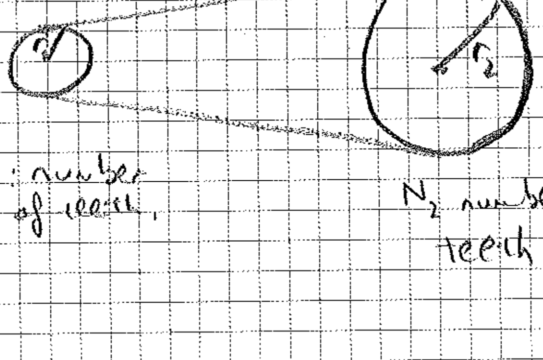
$$= r\frac{d\omega}{dt}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) + r\omega(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j})\frac{d\theta}{dt}$$

$$= r\alpha\underbrace{(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})}_{\text{Tangential direction}} + r\omega^2\underbrace{(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j})}_{\text{Radial Direction}}$$

$$\Rightarrow \boxed{a_{tan} = r\alpha} \qquad \boxed{a_{rad} = \omega^2 r = \frac{v^2}{r}}$$



Ex 9.5: How are the angular speeds of two bicycle wheels,



N_1: number of teeth, N_2 number of teeth

KOÇ ÜNİVERSİTESİ

110.5

The chain does not slip

$\Rightarrow$ The linear speed is the same for both wheels:

$\Rightarrow v_1 = v_2 = v \Rightarrow \omega_1 r_1 = \omega_2 r_2$

$$\Rightarrow \frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}$$

$2\pi r_1 = N_1 \times$ chain separation

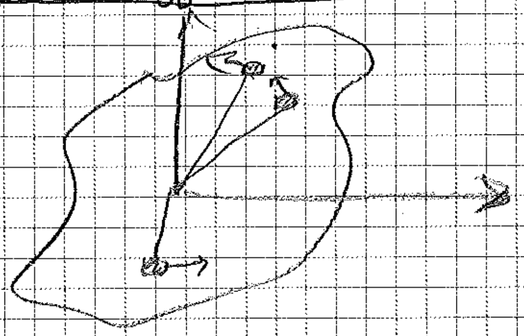
$2\pi r_2 = N_2 \times$ chain separation

$$\Rightarrow \frac{2\pi r_1}{N_1} = \frac{2\pi r_2}{N_2} \Rightarrow \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

$$\Rightarrow \boxed{\frac{\omega_1}{\omega_2} = \frac{N_2}{N_1}}$$

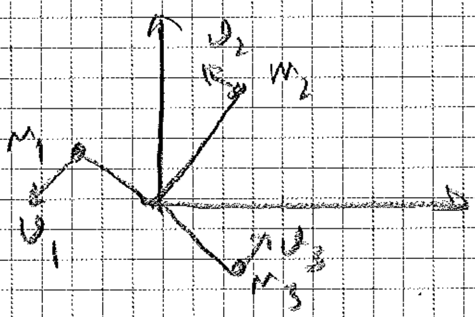
The bigger the number of chains the smaller angular speed.

9.4. Energy in Rotational Motion:

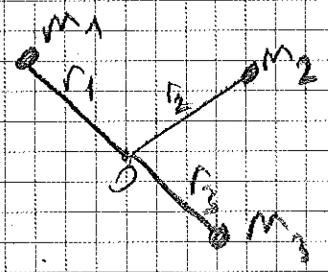


A rotating rigid body consists of moving masses, so there is a kinetic energy associated with rotational motion.

We will determine the rotational kinetic energy in terms of the properties of the rigid body and the angular velocity.



Consider the rigid body is made of point particles with masses m_1, m_2, m_3 located a distance r_1, r_2, r_3 away from the axis of rotation.



Total kinetic energy of the rigid body is:

$$K = K_1 + K_2 + K_3$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_1^2 + \frac{1}{2} m_3 v_3^2$$

$$= \underbrace{\left(\frac{1}{2} m_1 r_1^2 + \frac{1}{2} m_2 r_2^2 + \frac{1}{2} m_3 r_3^2\right)}_{\frac{1}{2} I} \omega^2$$

$$= \frac{1}{2} I \omega^2$$

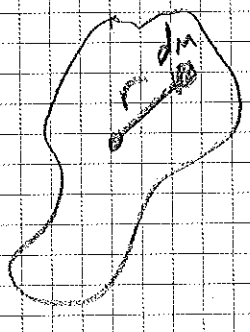
where $I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$ is called as the moment of inertia.

In general:

For a rigid body made of many point particles:

$$\boxed{I = \sum_i m_i r_i^2}$$

For a rigid body made of a distribution of masses:

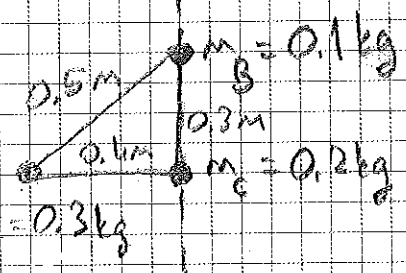


$$\boxed{I = \int r^2 dm}$$

* It is important to note that the value of I depends on both the rigid body and the rotation axis.

Examples:

Ex 9.7.: Consider a three body system:



a) What is the moment of inertia about an axis through point A?

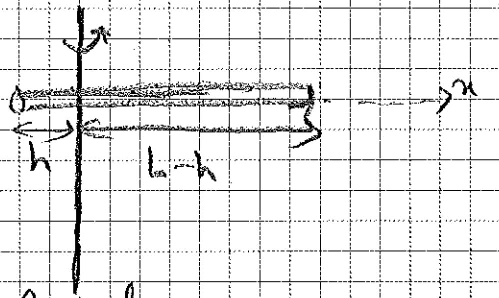
b) What is the moment of inertia about an axis coinciding with BC?

a) $I_A = m_A r_A^2 + m_B r_B^2 + m_C r_C^2 = 0.3\,kg \times 0^2 + 0.1 \times (0.5)^2 + 0.2 \times (0.4)^2$

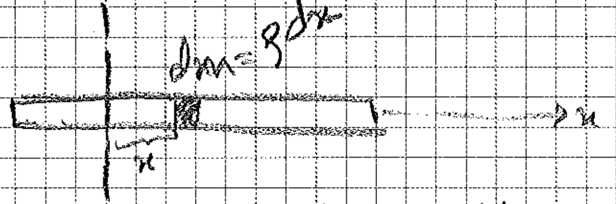
$= 0.025 + 0.032 = 0.057\ kg\,m^2$

b) $I_{BC} = 0.3 \times (0.4)^2 + 0 + 0 = 0.048\ kg\,m^2$

Ex 9.12:



Consider a rod with a uniform mass density. Total mass of the rod is M, and the length is L. What is I about the given axis of rotation?



ρ: mass density $= \frac{M}{L}$

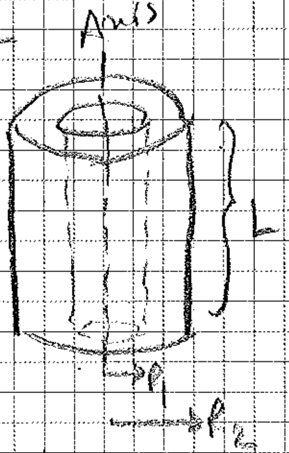
$$I = \int r^2 dm = \int_{-h}^{L-h} x^2 \rho\, dx$$

$$= \rho \frac{x^3}{3} = \frac{\rho}{3}\left((L-h)^3 - h^3\right)$$

$$= \frac{\rho}{3}\left\{L^3 - 3L^2 h + 3Lh^2\right\}$$

$$\Rightarrow I = \frac{1}{3}\,\frac{M}{L}\left\{L^3 - 3L^2 h + 3Lh^2\right\} = \boxed{\frac{1}{3}M\left\{L^2 - 3Lh + 3h^2\right\}}$$

Ex 9.13:



What is I about the axis of symmetry of the cylinder with a uniform mass density and total mass M?

$$I = \int_V r^2 dm$$

V: 3D integral

KOÇ ÜNİVERSİTESİ

Consider A thin sheet of thickness dr.

$$dI = dm\, r^2 \quad , \quad dm = \overset{\text{volumetric mass density}}{\rho} \cdot 2\pi r L\, dr$$

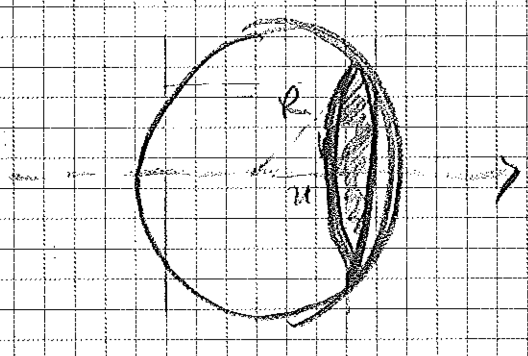
$$\Rightarrow I = \int_{r=R_1}^{R_2} dm\, r^2 = \int \rho 2\pi r L\, dr\, r^2 = 2\pi L \rho \int r^3 dr$$

$$= 2\pi L \rho \left. \frac{r^4}{4} \right|_{R_1}^{R_2} = 2\pi L \rho \left(\frac{R_2^4}{4} - \frac{R_1^4}{4} \right)$$

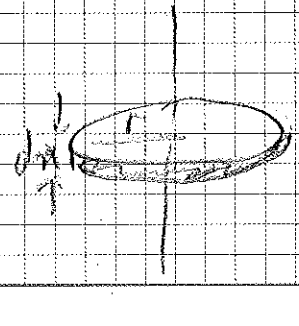
$$\rho = \frac{M}{\pi (R_2^2 - R_1^2) L} \quad \Rightarrow I = \frac{2\pi L M}{\pi (R_2^2 - R_1^2) L} \frac{1}{4} \left(R_2^4 - R_1^4 \right)$$

$$I = \boxed{\frac{M}{2} \left(R_2^2 + R_1^2 \right)}$$

Example 9.14:



Consider a solid sphere with a uniform mass density. What is I about an axis of rotation that goes through the center?



$$\rightarrow dI = \frac{dm}{2} r^2 = \frac{dm}{2} (R^2 - x^2)$$

$$dm = \rho \pi r^2 dx$$

KOÇ ÜNİVERSİTESİ

$$\Rightarrow dI = \frac{\rho\pi}{2}(R^2-x^2)\,dx\,(R^2-x^2) = \frac{\rho\pi}{2}(R^2-x^2)^2\,dx$$

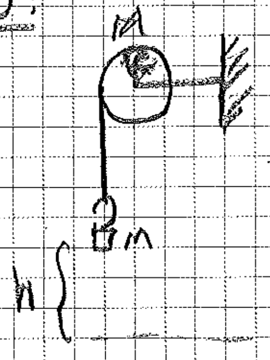
$$\Rightarrow I = \frac{\rho\pi}{2}\int_{-R}^{R}(R^2-x^2)^2\,dx = \frac{\rho\pi}{2}\int_{-R}^{R}(R^4-2R^2x^2+x^4)\,dx$$

$$= \frac{\rho\pi}{2}\left\{R^4(2R) - 2R^2\left(\frac{2R^3}{3}\right) + \frac{2R^5}{5}\right\}$$

$$= \frac{\rho\pi}{2}R^5\left\{2-\frac{4}{3}+\frac{2}{5}\right\} = \frac{\rho\pi}{2}R^5\left\{\frac{30-20+6}{15}\right\} = \frac{8\rho\pi}{15}R^5$$

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \Rightarrow I = \frac{\cancel{8}^{2}\rho\pi}{\cancel{15}_{5}}\cdot\frac{3MR^2}{\cancel{4}\pi} = \boxed{\frac{2}{5}MR^2}$$

Ex 9.9:



Consider a cylinder shaped pulley with negligible friction.
Object with mass m is released with no initial velocity at a distance h.
Speed of the object and the angular speed of the cylinder as the object hits the floor?

If the mass of the cylinder is ignored:

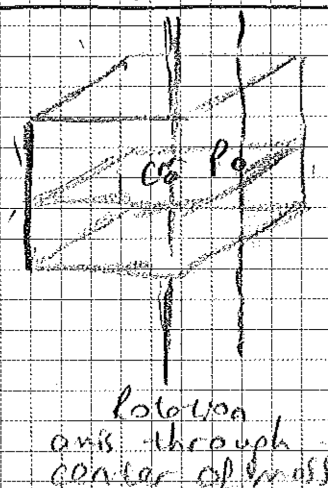
$$mgh = \frac{mv^2}{2} \Rightarrow v = \sqrt{2gh}$$

If M is not ignored:

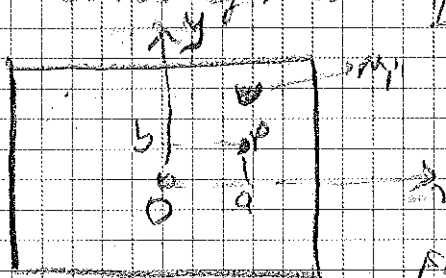
$$mgh = \frac{1}{2}I\omega^2 + \frac{mv^2}{2} = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{v^2}{R^2} + \frac{mv^2}{2}$$

$$\Rightarrow 2mgh = \left(\frac{M}{2}+m\right)v^2 \Rightarrow v = \sqrt{\frac{2mgh}{\frac{M}{2}+m}}$$

9.5. Parallel-Axis Theorem:



Consider a cube, and one axis of rotation that goes through the center of mass and another through a point P. parallel axis which passes through



If we place the cm to the origin of the coordinate system.

$$I_{cm} = \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2)$$

For the thin slice.

For the whole cube:

$$I_{cm} = \sum_{i,\,\text{cube}} m_i (x_i^2 + y_i^2)$$

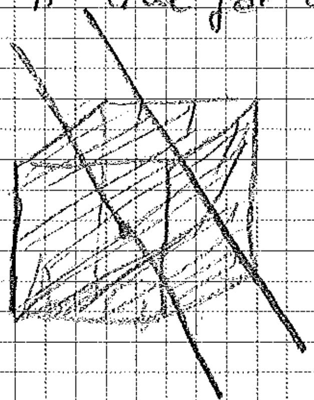
$$I_p = \sum_{i,\,\text{cube}} m_i \left((x_i - a)^2 + (y_i - b)^2 \right)$$

$$= \sum_i m_i (x_i^2 + y_i^2) - 2a \underbrace{\sum_i m_i x_i}_{=0} - 2b \underbrace{\sum_i m_i y_i}_{=0} + \sum_i m_i (a^2 + b^2)$$

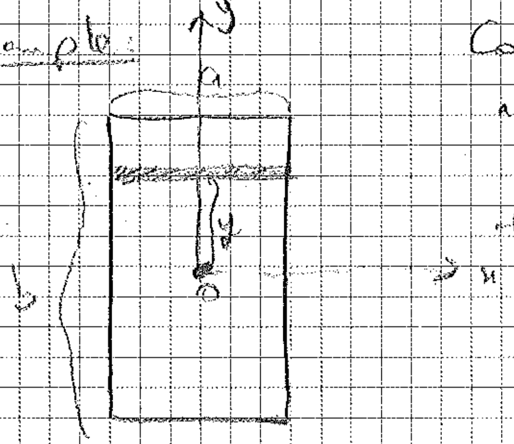
$$\boxed{I_p = I_{cm} + Md^2}$$

distance between the two axis of rotation.

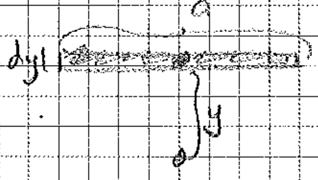
This is true for any axis of rotation



Example:



Consider a thin rectangular sheet with a uniform mass density and a total mass M. What is I for a axis of rotation through the cm?



$$\frac{1}{12}(\rho a\,dy)a^2$$

$$dI_0 = \frac{1}{12}\rho a^3 dy + \rho a\,dy\,y^2$$

$$\Rightarrow I_0 = \int_{-b/2}^{b/2}\left(\frac{1}{12}\rho a^3 dy + \rho a y^2 dy\right) = \frac{1}{12}\rho a^3 b + \rho a\frac{2}{3}\frac{b^3}{8}$$

$$= \frac{1}{12}\rho ab\left(a^2+b^2\right) = \frac{1}{12}M\left(a^2+b^2\right)$$