Undergrad Exam Solution Fall 2019

a) Sol. Both linear and angular momenta are remain conserved before and after collision.

b) Sol. The new center of mass after collision $\frac{m\frac{L}{2}+mL}{2m}=\frac{3L}{4}$

The moment of inertia of combined object (point mass + rod) around its center of mass can be evaluated as; $I_{cm}=(m\frac{L^2}{12}+m\frac{L^2}{16})+m\frac{L^2}{16} \Rightarrow = m\frac{L^2}{12}+m\frac{L^2}{8}=\frac{5mL^2}{24}$.

c) Sol. By law of conservation of linear momentum

$$P_i=P_f$$

$$mv_0=2mv_{cm} \Rightarrow v_{cm}=v_0/2$$ (linear center of mass velocity)

And angular momentum conservation law

$$L_i=L_f$$

$$mv_0\frac{L}{4}=I_{cm}\omega$$

$$mv_0\frac{L}{4}=\frac{5mL^2}{24}\omega \quad \text{where } I_{cm}=\frac{5mL^2}{24}$$

$$\omega=\frac{6v_0}{5L}$$ required angular velocity and its direction is out of page!

d) Sol. The kinetic energy of the system

$$K.E=\frac{1}{2}(2m\,v_{cm}^2)+\frac{1}{2}\left(\frac{5mL^2}{24}\right)\left(\frac{6v_0}{5L}\right)^2$$

$$\Rightarrow \frac{1}{2}m\left[2\left(\frac{v_0}{2}\right)^2+\frac{3}{10}v_0^2\right]$$

$$\Rightarrow \frac{1}{2}m\left[\frac{1}{2}v_0^2+\frac{3}{10}v_0^2\right]$$

$$K.E=\frac{2}{5}mv_0^2$$

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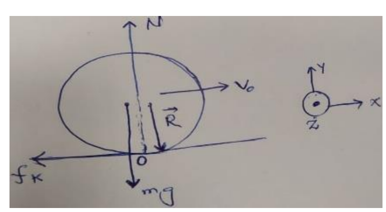
Q. 2 a) Sol. From figure, we can write



$$\vec{V}(t=0)=v_0\,\hat{x}$$

$$\vec{\omega}(t=0)=-\omega_0\hat{z}$$

$v_0 > \omega_0$R => $\vec{f}_k$=-$f_k\hat{x}$ (at touching point to ground)

where f_k=μ_k.N

b) Sol. $\hat{y}$: N-mg= ma_y since a_y=0

=> N-mg=0 => N=mg

$\hat{x}$: $-f_k$=ma_x => -$\boldsymbol{\mu}$mg=ma_x

a_x = -$\boldsymbol{\mu}$g

$\hat{z}$: -Rf_k=I$\boldsymbol{\alpha}$=> -$\boldsymbol{\mu}$Rmg =$\frac{2}{5}mR^2\alpha$ $\boldsymbol{\alpha}$=-$\frac{5\boldsymbol{\mu}}{2R}g$

Center of mass speed;

$\hat{v}_x$(t)= ($v_0 -$ μgt) $\hat{x}$

$\vec{V}$(t)= $\vec{\omega}$(t) × $\vec{R}$

=> =(- ω_0- $\frac{5\boldsymbol{\mu}}{2R}g$t) $\hat{z}$ ×(-R) $\hat{y}$

=> =-($v_0 + \frac{5\boldsymbol{\mu}}{2}g$t) $\hat{x}$

Total speed of the body at touching point;

$\hat{v}_x$(t)+ $\vec{V}$(t)= v_0- ω_0R-$\frac{7\boldsymbol{\mu}}{2}g$t (a)

At t=T, we have rolling without slipping, therefore, the total velocity is 0. Eq. (a) becomes

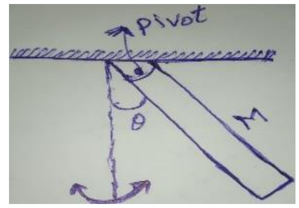
0= v_0- ω_0R-$\frac{7\boldsymbol{\mu}}{2}g$T

=> v_0- ω_0R=$\frac{7\boldsymbol{\mu}}{2}g$T

T= $\frac{2}{7}\frac{(v_0 - \omega_0 \text{R})}{\mu g}$

Q. 3 a) Sol. The moment of inertia of the rod at pivot point.

I_{cm}=$\frac{1}{12}Ml^2 + \frac{1}{4}Ml^2$



$=> \; I_{cm} = \frac{1}{3} M l^2$

The torque produced by the rod during its motion

τ= r×$\vec{F}$ => **τ**= r$\vec{F}$Sinθ

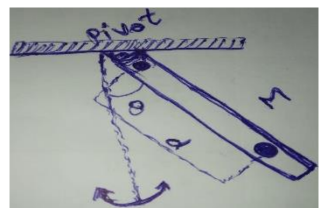
τ= - $\frac{l}{2}$Mg$\sin\theta$ = $\frac{1}{3} M l^2 . \frac{d\theta^2}{dt^2}$

For small deflection i.e. $\sin\theta \approx \theta$

$\frac{d\theta^2}{dt^2} + \frac{3g}{2l}\theta = 0.$ => $\ddot{\theta} = \omega^2\theta$ ω=$\sqrt{\frac{3g}{2l}}$

Required equation of motion, which is similar to simple harmonic motion.

b) Sol. The total inertia of the rod is



$I_{total} = \frac{1}{3} M l^2 + m d^2$

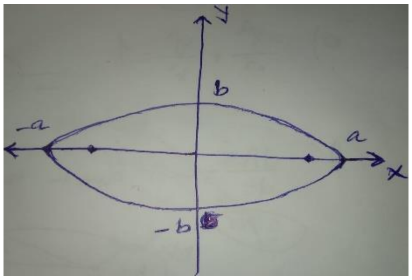
Torque of the rod

τ= -$\frac{l}{2}$$Mg\sin\theta$ -$mgd\sin\theta$

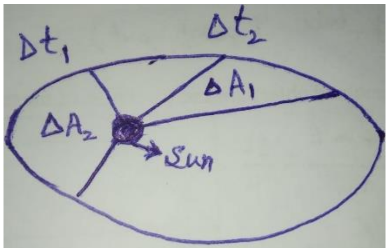
$I_{total} . \frac{d\theta^2}{dt^2}$ = - $\frac{l}{2}$$Mg\sin\theta$ -$mgdSin\theta$ **τ**=$I_{total} . \frac{d\theta^2}{dt^2}$

- ⇨ $\frac{d\theta^2}{dt^2} + \frac{Mgl/2 + mgd}{\frac{1}{3}Ml^2 + md^2}\theta = 0$
- ⇨ $\ddot{\theta} = \omega^2\theta$ ω=$\sqrt{\frac{Mgl/2 + mgd}{\frac{1}{3}Ml^2 + md^2}}$
- ⇨ $\frac{3g}{2l} => 2l = 3d$ => d = $\frac{2l}{3}$ is the point at which the frequency of the oscillation remains the same.

Q.4 a) Sol. $Equation \;\; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



b) Sol. $\frac{dA}{dt}$ **=** Constant due to the conservation of angular moment (Kepler's 2nd law)



i.e. $\Delta A_1 = \Delta A_2$ if $\Delta t_1 = \Delta t_2$

c) Sol. $\frac{a^3}{T^2}$ = Constant (Kepler's 3rd law)

Proof. The centripetal force of the planet balance the gravitation force i.e

$$\vec{F}_c = \vec{F}_g$$

$$\frac{mv^2}{r} = \frac{GmM}{r^2}$$

$$v = \sqrt{\frac{GM}{r}}$$

Since $v = r\omega = \frac{2\pi r}{T}$ $\qquad \omega = \frac{2\pi}{T}$

$$\frac{2\pi r}{T} = \sqrt{\frac{GM}{r}} \Rightarrow (2\pi r/T)^2 = \frac{GM}{r}$$

$$T^2 = \frac{4\pi^2}{GM} r^3 \Rightarrow \frac{r^3}{T^2} = \frac{GM}{4\pi^2} = \text{constant}$$