Q. You wind a wire around your 10 cm long pen which has a circular cross sectional area of 1 $\text{cm}^2$. You start winding from the tip of the pen to the end and continue from the end back to the tip again. You make 100 turns in each round. Find the self-inductance of the resulting solenoid if you wind the wire

(a) clockwise at all times.

(b) clockwise in the first pass and counter-clockwise on the way back.

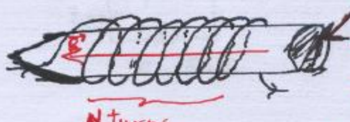
Hint: The magnetic field inside a standard solenoid with $n$ turns per unit length is $B = \mu_0 n i$. The emf induced across an inductor is $\mathcal{E} = L\frac{di}{dt}$.

We basically find the flux and take it, derivative with respect to time to get $|\mathcal{E}| = |-\frac{d\Phi_B}{dt}|$.

(a)

$$\Phi_B = \int \vec{B}\cdot\hat{n}\,ds$$

First Round




a = The area of Cross section of the pen

N turns

$n = \frac{100 \text{ turns}}{0.1 \text{ m}} = 10^3$ turns per unit length (m)

$$\Phi_B = \mu_0 n i a + \mu_0 n i a = 2\mu_0 n i a$$

Note that both magnetic fields are in the same direction; because, the turns in the first and second round are both clockwise.


$\vec{B} = \vec{B}_1 + \vec{B}_2$

due to first round

due to second round

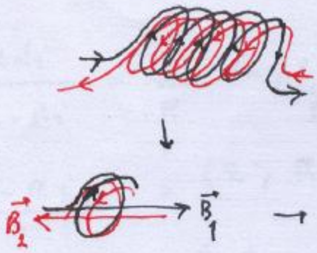
Now taking the derivative of $\Phi_B$ we have:

$$|\mathcal{E}| = \left|-\frac{d\Phi_B}{dt}\right| = \left|-2\mu_0 n a \frac{di}{dt}\right| = 2\mu_0 n a \left|\frac{di}{dt}\right|$$

So according to $|\mathcal{E}| = L\left|\frac{di}{dt}\right|$ we get:

$$L = 2\mu_0 n a$$

(b) As it is shown when the second round is counte-clockwise, the produce magnetic field is in the opposite direction and since they are equal in magnitude, they cancel each other. Thus $\vec{B} = 0$.




$\rightarrow \vec{B} = \vec{B}_1 + \vec{B}_2$

Since $\vec{B} = 0$, there is no flux.

Therefore from $|\mathcal{E}| = L\left|\frac{di}{dt}\right| \longrightarrow L = 0$.

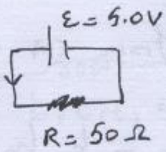
Q. A 5.0 V battery, a 50 Ω resistance and a 2.0 mH inductor are all connected in series with an open switch. The switch is suddenly closed.

(a) What is the final value of the energy stored in the inductor?

(b) How long after closing the switch will the energy stored in the inductor reach one-half of its final value?

Hint: For a charging inductor, $i_L(t) = I_f\,(1 - e^{-t/\tau})$.

(a) After a long time we replace the inductor with a conductor




So $I_f = \dfrac{\mathcal{E}}{R} = \dfrac{5.0}{50} = 0.10$ A.

$E_f = \frac{1}{2} L I_f^2 = \frac{1}{2} \times 2.0 \times 10^{-3} \times (0.10)^2$

$= 1.0 \times 10^{-5}$ J

(b) At each moment:

$E_{(t)} = \frac{1}{2} L i^2_{(t)}$

we also have $E_f = \frac{1}{2} L I_f^2$; therefore,

$$\frac{E_{(t)}}{E_f} = \frac{i^2_{(t)}}{I_f^2} = \frac{I_f^2 (1 - e^{-t/\tau})^2}{I_f^2} = (1 - e^{-t/\tau})^2 \qquad (1)$$

we ~~also~~ want $t_0$ at which we have:

$$\frac{E_{(t_0)}}{E_f} = \frac{1}{2} \xRightarrow{(1)} (1 - e^{-t_0/\tau})^2 = \frac{1}{2} \Rightarrow 1 - \frac{\sqrt{2}}{2} = e^{-t_0/\tau}$$

$$\Rightarrow t_0 = -\tau \ln\left(1 - \frac{\sqrt{2}}{2}\right) \qquad (2)$$

So $\tau = \dfrac{L}{R}$. (3)

Now from (1) and (3) we will have:

$$t_0 = -\frac{L}{R} \ln\left(1 - \frac{\sqrt{2}}{2}\right) = -\frac{2.0 \times 10^{-3}}{50} \overbrace{\ln\left(1 - \frac{\sqrt{2}}{2}\right)}^{\simeq 1.23} = 4.9 \times 10^{-5} \text{ s}$$

Q. A capacitor with capacitance $2.0\,\mu$F is initially charged with $20\,\mu$C. It is then connected to an inductor with an inductance of 20.0 mH.

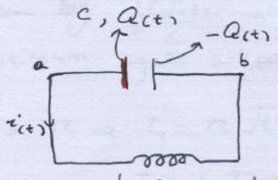
(a) What is the period of charge/current oscillations in the circuit?

(b) How long after the connection will the charge on the capacitor be zero?

(c) How long after the connection will the energy stored in the capacitor be maximum again?

3 (a) We have

$$V_{ab} = \frac{Q(t)}{C} \quad (1)$$




$L, \mathcal{E}(t) = L \frac{di(t)}{dt}$

Also

$$V_{ab} = \mathcal{E}(t) = L \frac{d}{dt} i(t) \quad (2)$$

To relate $i(t)$ to $Q(t)$ note that if, e.g., $Q(t)$ is positive, then $i(t)$ is in the direction shown. But since $Q(t)$ will be decreasing so $\frac{dQ(t)}{dt}$ is negative. This means that if we take the positive direction of current in the direction shown, we have to

$$i(t) = -\frac{dQ(t)}{dt} \quad (3)$$

Thus, from (1) and (2), and (3) it follows:

$$\frac{Q(t)}{C} = -L\frac{d^2Q(t)}{dt^2} \Rightarrow Q(t) = Q_0 \cos\left(\frac{t}{\sqrt{LC}}\right)$$

where $Q_0$ is the initial charge on the capacitor ($Q(t=0)$).

We see the frequency of oscillations is $\omega = \frac{1}{\sqrt{LC}}$.

Then,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC} = \overset{6.28}{2\times 3.14}\sqrt{20.0\times10^{-3}\times 2.0\times10^{-6}} = 12.56\times10^{-4}\,s \simeq 1.3\ ms$$

(b) We want $t_0$ that satisfies (the first time that $Q(t)$ is zero)

$$Q(t_0) = Q_0 \cos\left(\frac{t_0}{\sqrt{LC}}\right) = 0 \Rightarrow \frac{t_0}{\sqrt{LC}} = \frac{\pi}{2} \Rightarrow t_0 = \frac{\pi}{2}\sqrt{LC}$$

$$t_0 = \frac{1}{4}T \approx 3.1\ ms$$

(c) Energy in the capacitor is given by $\frac{1}{2}\frac{Q^2(t)}{C}$ at each moment. Therefore, it will be maximum again when $Q^2(t_1) = Q_0^2$.

$$\text{so } Q^2_{(t)} = Q_0^2 \cos^2\left(\frac{t_1}{\sqrt{LC}}\right) = Q_0^2 \Rightarrow \frac{t_1}{\sqrt{LC}} = \pi \Rightarrow t_1 = \pi\sqrt{LC}$$

$$\Rightarrow t_1 = \frac{1}{2}T \approx 6.3\ ms$$

Q. A solenoid with its axis along the x-direction has a cross-sectional area $A$ and length $L$. It has nonuniform windings resulting in a magnetic field $B(x) = \mu_0 n i[(1 + 0.2\cos(2\pi x/L)]$, where $n$ is the average winding number per unit length, $i$ is the current.

(a) Find the total energy stored in the magnetic field.

(b) Find the self-inductance of the solenoid by equating the total energy to $U = Li^2/2$.

(a) The density of magnetic field energy is given by

$$u = \frac{1}{2}\frac{B^2}{\mu}$$

at each point of space, at each moment.

Here $\mu = \mu_0$ and then the total energy stored in the magnetic field of a solenoid will be given by

$$U = \frac{1}{2\mu_0}\int_{\substack{\text{inside the}\\ \text{Solenoid}}} B^2\, dV = \frac{1}{2\mu_0}\int_0^L B^2_{(x)} A\, dx = \frac{A}{2\mu_0}\int_0^L \left[\mu_0 n i\left(1 + 0.2\cos\left(\frac{2\pi x}{L}\right)\right)\right]^2$$

$$\Rightarrow U = \frac{1}{2}A\mu_0 n^2 i^2 \int_0^L \left(1 + 0.2\cos\left(\frac{2\pi x}{L}\right)\right)^2 dx$$

$$= \frac{1}{2}A\mu_0 n^2 i^2 \int_0^L \left[1 + 0.04\cos^2\left(\frac{2\pi x}{L}\right) + 0.4\cos\left(\frac{2\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2}A\mu_0 n^2 i^2 \left[L + 0.04\int_0^L \cos^2\left(\frac{2\pi x}{L}\right)dx + \cancel{(+0.4)\sin\left(\frac{2\pi}{L}x\right)\Big|_0^L}\right]$$

$$= \frac{1}{2}A\mu_0 n^2 i^2 \left[L + \frac{0.04}{2}\int_0^L \left(1 + \cos\left(\frac{4\pi x}{L}\right)\right)dx\right]$$

$$= \frac{1}{2}A\mu_0 n^2 i^2 \left[L + 0.02L + \cancel{\frac{0.04}{2}\int_0^L \cos\left(\frac{4\pi}{L}x\right)dx}\right]$$

$$U = 0.56\, A\mu_0 n^2 i^2 L \qquad (1)$$

(b) From (1) and using $U = \frac{1}{2}L_{ind.}i^2$ we'll have:

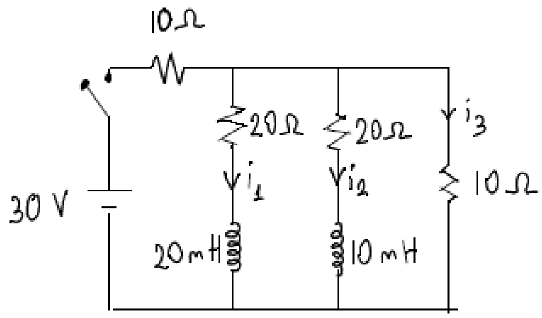
$$\overset{1.02}{\cancel{0.56}}\, A\mu_0 n^2 \cancel{i^2} L = \frac{1}{\cancel{2}} L_{ind.} \cancel{i^2} \Rightarrow L_{ind.} = 1.02\, A\mu_0 n^2 L$$

Q. The switch in the circuit below is closed at $t = 0$.

(a) What are the values of $i_1$, $i_2$, $i_3$ immediately after the switch is closed?

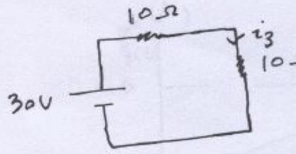
(b) What are the values of $i_1$, $i_2$, $i_3$ long after the switch is closed?

(c) Sketch $i_1$, $i_2$, $i_3$ as a function of time.




5 (a) At the initial moment inductors act like open switches. So we are dealing with an equivalent circuit as below

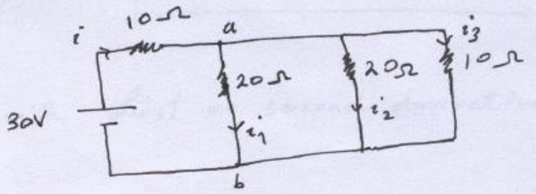
with $i_1 = i_2 = 0$.




Thus:

$$i_3 = \frac{30}{10+10} = 1.5 \text{ A}$$

(b) After a long time (comparing to the time constant of the switch) we will get an equivalent circuit with the inductors replaced by conductors. The reason is that the current reaches to a constant valu in each branch. Thus emf produced by each inductor, which is given by $\mathcal{E} = L\frac{di}{dt}$ will be zero and the inductor behaves as a piece of wire. Therefore, the equivalent circuit is as below:



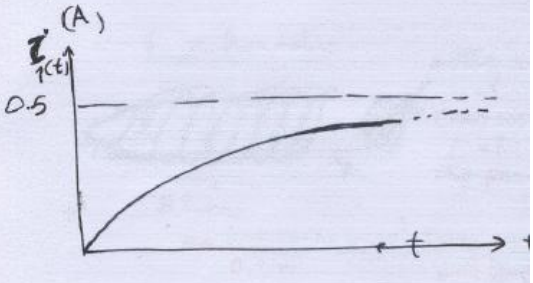

The total resistance is given by:

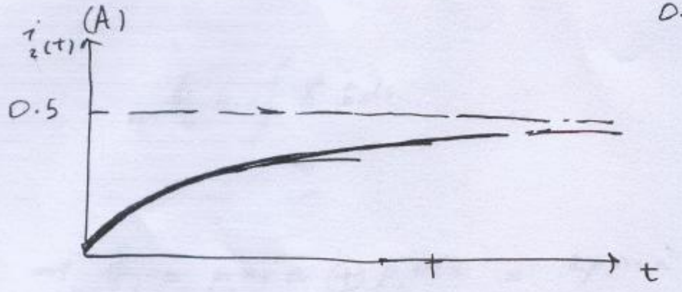
$$R_T = 10 + \frac{1}{\underbrace{\frac{1}{20} + \frac{1}{20}}_{2/10} + \cancel{\frac{1}{10}}} = 15\,\Omega$$

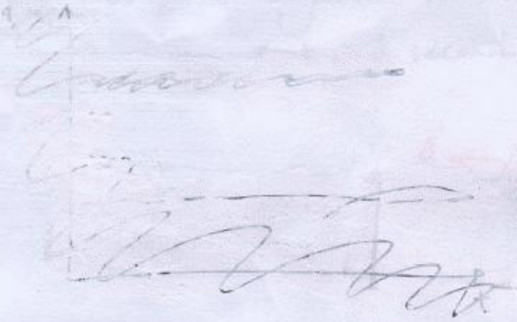
$$\Rightarrow i = \frac{30\text{V}}{15\,\Omega} = 2 \text{ A}$$

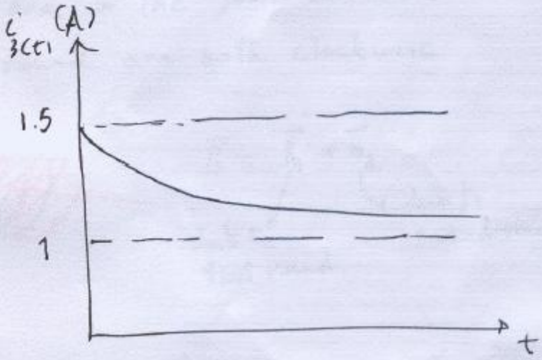
(C) $i_1 = i_2 = 0$ at $t = 0$

$\begin{cases} i_1 \longrightarrow 0.5\text{ A} \\ i_2 \longrightarrow 0.5\text{ A} \end{cases}$ at infinite time.






Note that there is no points with first or second derivatives zero except after infinite time.