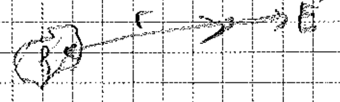
Chapter 22: Gauss' Law



$$\vec{E} = \underbrace{\int k \frac{dq}{r^2}\, d\hat{r}}_{\text{over the source}}$$

If the source object has a certain symmetry property we can find the electric field in an easier way than applying the integration. We will apply the Gauss' Law.

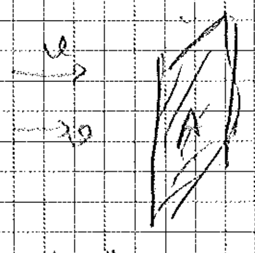
Consider a certain distribution of charge surrounded by an imaginary surface.

Gauss' Law gives the relationship between the field at all points on the surface and the total charge enclosed.

We have to first define the electric flux.

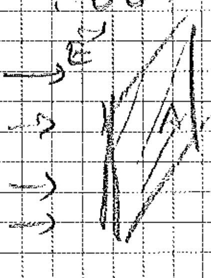
Electric Flux:

In the case of a fluid flow:



the flux is vA, for flow perpendicular to the surface

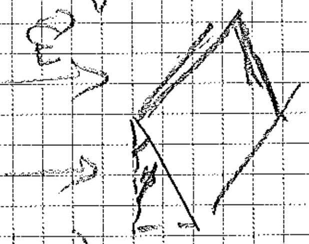
In analogy, we define the electric flux:



$\Phi_E = EA$ for constant incident electric field in the direction of the surface normal.

Electric Flux: The Amount of the electric field lines passing through an area.

For the case of uniform electric field not parallel to the surface normal:



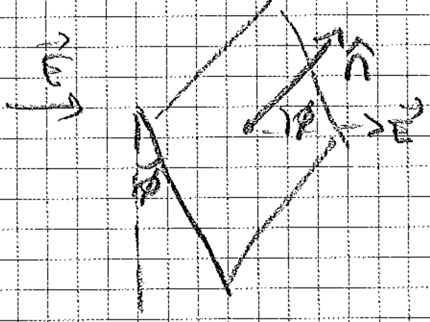
$$\Phi_E = E\underbrace{A\cos\phi}$$

area of the projection plane.

Another way to interpret $\Phi_E = EA\cos\phi$ is by the definition of the "area vector"

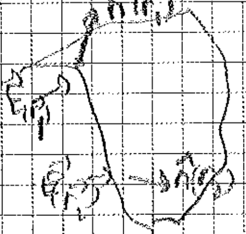
$$\vec{A} = A\,\hat{n}$$

area — unit vector along the surface normal (perpendicular to the plane)



$$\Rightarrow \boxed{\Phi_E = \vec{E}\cdot\vec{A}}$$

In general, for the case of a nonuniform electric field, and curved surface:

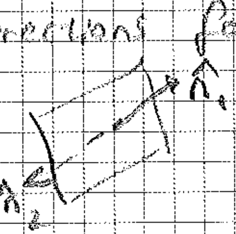


$$\Rightarrow \boxed{\Phi_E = \int_{\text{over the surface}} \vec{E}(\vec{r})\cdot d\vec{A}(\vec{r})}$$

This is a surface integral.

$d\vec{A} = dA\,\hat{n}$, infinitesimal area element

Remark: It is important to note that there are two possible directions for $\hat{n}$.



$\hat{n}_1 = -\hat{n}_2$

outward — inward

One will denote the outward flux and the other the inward flux.

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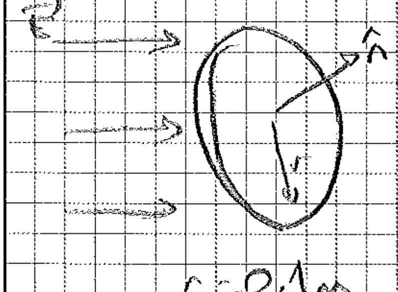
Examples:

Ex 22.1: A disk is oriented with its normal unit vector at an angle 30° to a uniform electric field $\vec{E} = 2\times10^3$ N/C.



$r = 0.1$ m

a) What is the E-flux through the disk

$$\Phi_E = 2\times10^3 \times \pi \times 10^{-2} \times \frac{\sqrt{3}}{2}$$

$$= 54 \ \mathrm{Nm^2/C}$$

b.) What is the flux through the disk if it is turned so that its normal is perpendicular to $\vec{E}$?

$$\Phi_E = 0$$

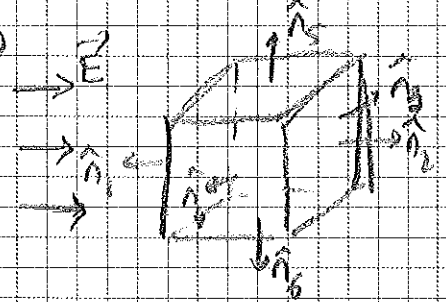
Ex 22.2: A cube of side L is placed in a region of uniform Electric field, $\vec{E}$.

What is the flux through each face of the cube and the total flux through the cube when

a) it is oriented perpendicular to the cube

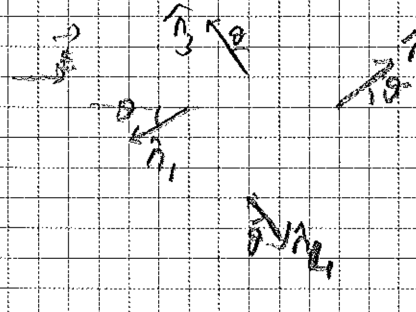
b) the cube is turned by an angle θ.

a)



outward flux: $\Phi_E = \overbrace{-EL^2}^{\text{surface 1}} + \overbrace{EL^2}^{\text{surface 2}}$
$+0+0+0+0$
$= 0$

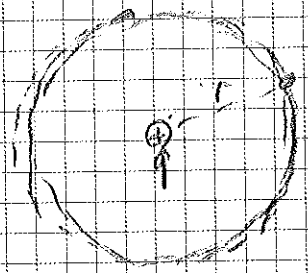
b)



outward flux:

$$\Phi_E = -\cancel{EL^2\cos\theta} + \cancel{EL^2\cos\theta} + EL^2\cos\left(\frac{\pi}{2}+\theta\right)$$
$$+ EL^2\cos\left(\frac{\pi}{2}-\theta\right) + 0 + 0$$
$$= -\cancel{EL^2\sin\theta} + \cancel{EL^2\sin\theta} = 0$$

Ex 22.3: A positive point charge $q = 3\mu C$ is surrounded by a sphere with radius $0.2m$ centered on the charge.
What is the electric flux through the sphere?



$$\Phi_E = \int_{\substack{\text{surface} \\ \text{of the sphere}}} \vec{E} \cdot d\vec{A}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{e}_r \quad , \quad d\vec{A} = dA\,\hat{n} = dA\,\hat{e}_r$$

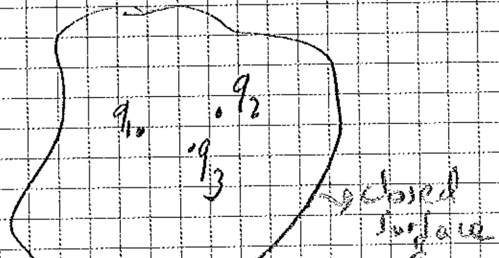
normal direction = radial.

$$\Rightarrow \Phi_E = \int \underbrace{\left(\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}\right)}_{\text{constant over the surface}} dA = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \underbrace{\int dA}_{\text{area of the sphere}} = \frac{1}{\cancel{4\pi}\varepsilon_0} \frac{q}{\cancel{r^2}} \cancel{4\pi r^2} = \frac{q}{\varepsilon_0}$$

independent of r as it should be!

22.3. Gauss' Law:

Statement of the law:
"Total outward electric flux through any closed surface is proportional to the net electric charge enclosed by the surface."

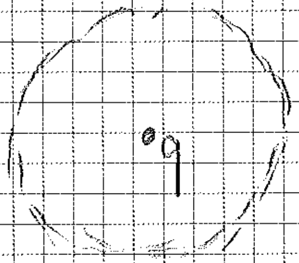


$$\boxed{\int_S \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}}$$

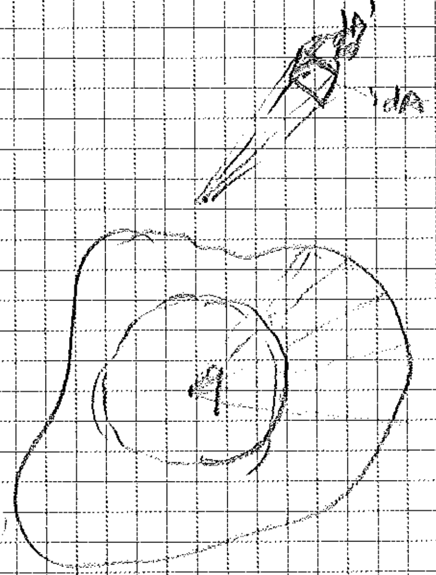
Q: total charge enclosed
$= q_1 + q_2 + q_3 \ldots$

Proof: For a point charge q located at the center of a spherical surface:

$$\Phi_E = \frac{q}{\varepsilon_0}$$



We can use the projection technique to go from the spherical surface to any arbitrary surface enclosing the charge q.



dA can be substituted by dA' such that $\vec{E}\cdot d\vec{A} = \vec{E}\cdot d\vec{A}'$

$$\rightarrow \Phi_E = \int_{\substack{\text{any surface}\\ \text{enclosing}\\ \text{the charge } q}} \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

If a surface encloses more than one charge:

$$\Phi_E = \int \vec{E}\cdot d\vec{A} = \int(\underbrace{\vec{E}_1}_{\substack{\text{electric}\\ \text{field due}\\ \text{to } q_1}} + \underbrace{\vec{E}_2}_{\substack{\text{Elec.}\\ \text{field}\\ \text{due to } q_2}})\cdot d\vec{A} = \int \vec{E}_1\cdot d\vec{A} + \int \vec{E}_2\cdot d\vec{A} = \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0}$$

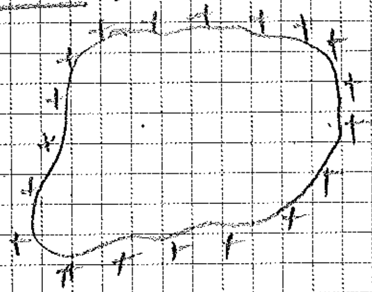
$$= \frac{Q}{\varepsilon_0}, \quad Q = q_1 + q_2 \qquad \blacksquare$$

22.4. Applications of Gauss' Law:

[We will choose a Gaussian surface in accordance with the symmetry of the problem and find the electric field.]

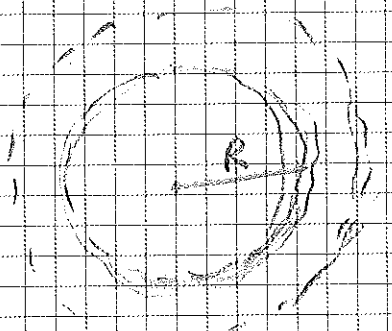
Ex 22.5: We place positive charge q on an ideal solid conducting sphere with radius R. Find $\vec{E}$ at any point inside and outside the sphere.

Remark: Conductors.



Net charge placed on an ideal solid conductor will stay on the surface of the conductor at steady state.
$\Rightarrow$ Electric field inside a conductor is 0.

If there was a non zero electric field inside the conductor a force would be applied to charges $\Rightarrow$ charges would move.
Since we know it is steady state $\Rightarrow$ charges do not move.
All the charges inside should go to the surface and electric field inside is therefore 0.



At any point inside the conductor:

$$E(r) = 0 \quad , \text{for} \quad 0 < r \leq R$$

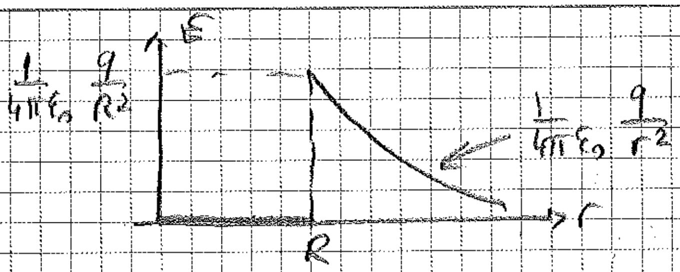
At any point outside the conductor:
Use Gauss' law applied to a spherical surface with radius $r > R$

$$\oint \vec{E} \cdot d\vec{A} = E\, 4\pi r^2 = \frac{q}{\varepsilon_0} \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

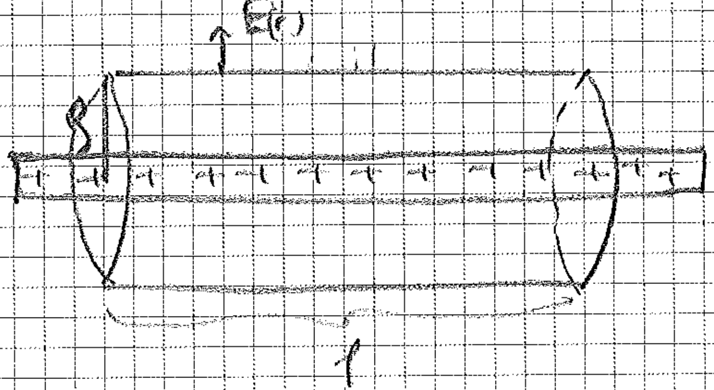
constant and in the radial direction because of the spherical symmetry

$$\Rightarrow \boxed{\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{e}_r}$$

for $r > R$



Ex 22.6: Consider an infinitely long, thin wire in which electric charge is uniformly distributed with a charge density λ. What is the electric field?



The problem has cylindrical symmetry, and the field lines should only have radial components.

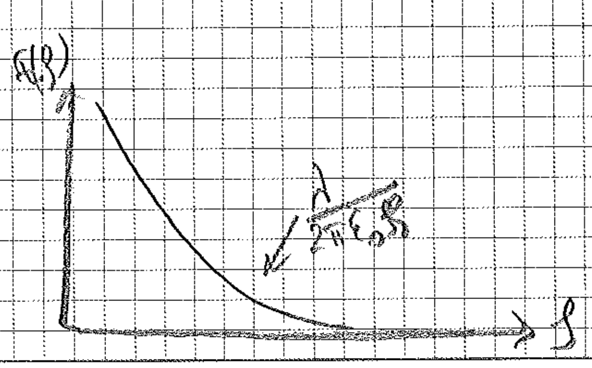
Field magnitude should be the same at all points with the same radial distance r from the wire.

⇒ We choose a cylindrical closed Gaussian surface with a length ℓ.

$$\Rightarrow \oint \vec{E}\cdot d\vec{A} = 2\pi\rho\cancel{\ell}E(\rho) = \frac{\lambda\cancel{\ell}}{\epsilon_0} \Rightarrow E(\rho) = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{\rho}$$

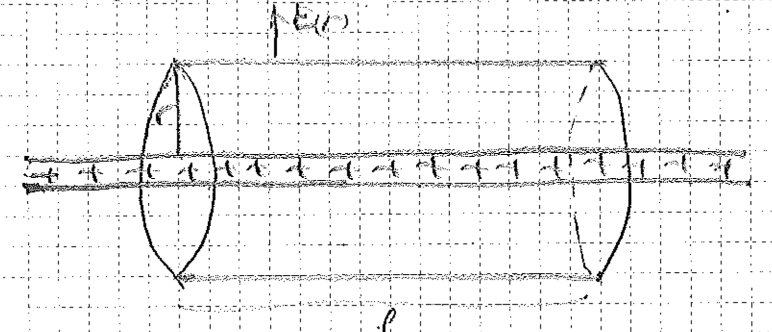
$$\Rightarrow \boxed{\vec{E}(\rho) = \frac{\lambda}{2\pi\epsilon_0\rho}\hat{e}_\rho}$$

radial direction in cylindrical coordinates



Ex. 22.6: Field of a line charge

Electric charge is distributed uniformly along an infinitely long, thin wire, having a charge density λ ⇒ What is the electric field?



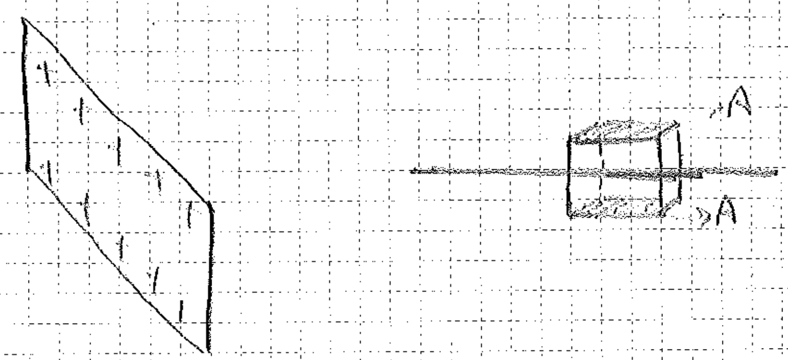
The problem has cylindrical symmetry, and the field lines should only have radial components.

Field magnitude should be the same at all points with the same radial distance, r, from the wire.

Choose a cylindrically, closed Gaussian surface with a length l.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = E(r)\, 2\pi r l = \frac{\lambda l}{\epsilon_0} \Rightarrow \boxed{E(r) = \frac{\lambda}{2\pi \epsilon_0 r}}$$

Ex 22.7: Electric field due to a thin, flat, uniform sheet with a positive charge per unit area of σ.

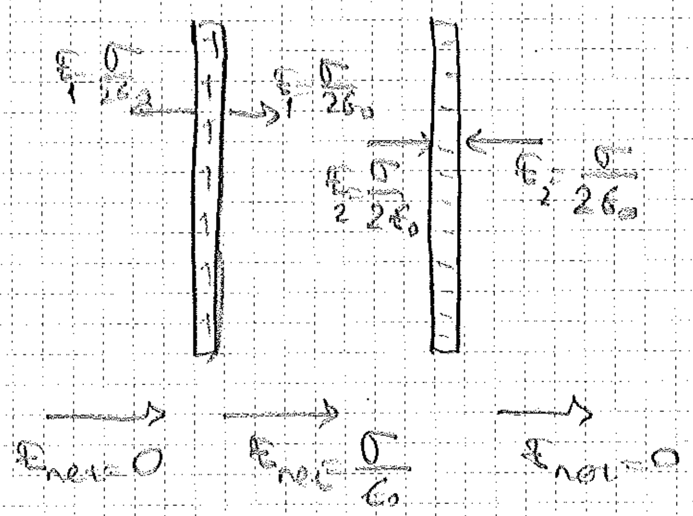


Due to symmetry, the field lines should be in the vertical direction.

Consider a cubic Gaussian surface which has the conducting sheet in the middle.

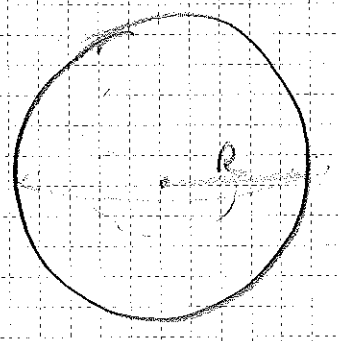
$$\Rightarrow \quad \Phi_E = 2EA = \frac{\sigma A}{\epsilon_0} \Rightarrow \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

E is independent from the distance from the conducting sheet



Ex 22.9: Field of a uniformly charged sphere.

Positive electric charge Q is uniformly distributed throughout the volume of an insulating sphere with radius R. What is the electric field?



The problem has spherical symmetry.
$\Rightarrow$ E field will be in the radial direction and magnitude of the E field will be the same at all points with the same distance r from the center of the insulating sphere.

Inside the insulator: $\oint \vec{E}(r) \cdot d\vec{A} = E(r) 4\pi r^2 = \frac{1}{\epsilon_0} \frac{Q}{\frac{4}{3}\pi R^3} \frac{4}{3}\pi r^3$

$$\Rightarrow \boxed{E(r) = \frac{Q}{4\pi\epsilon_0 R^3} r}$$

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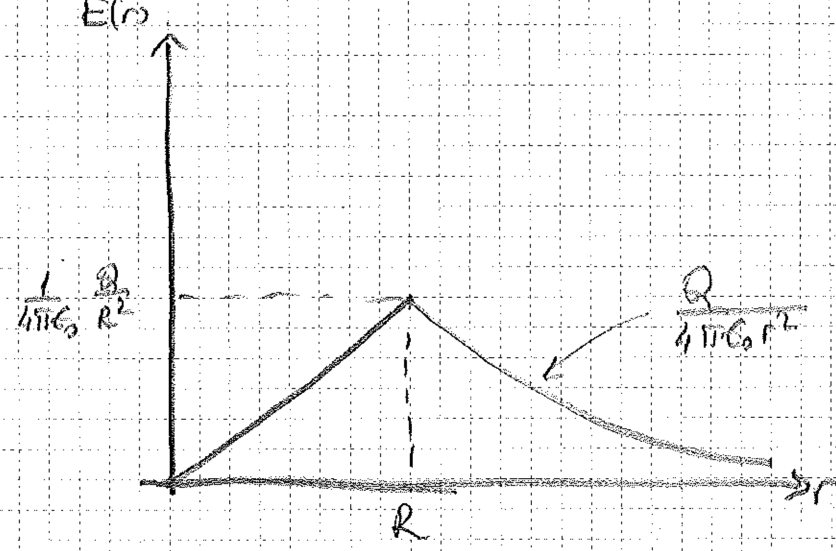
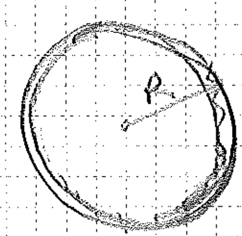
Outside the sphere: $r \geq R$

$\Rightarrow \oint \vec{E} \cdot d\vec{A} = E(r)\, 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow \boxed{E(r) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2}}$



Ex 22.10: Field of a hollow charged sphere.

The electric field due to a thin-walled sphere.



In side the sphere $E = 0$

Outside the sphere:

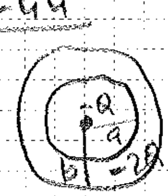
Problem has spherical symmetry $\Rightarrow$ E-field is in the radial direction with constant magn. for const. r.

$\Rightarrow$ Choose a spherical Gaussian surface:

$E(r)\, 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow \boxed{E(r) = \frac{Q}{4\pi\epsilon_0 r^2}}$

Prob. 22.44

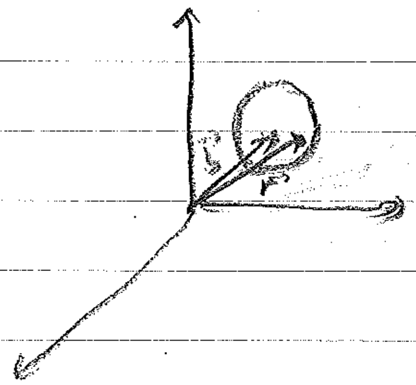
Ex:



- Electric field as a function of r
- Surface charge density at r=a and r=b

Prob 22.61. a) Insulating sphere ρ, charge density
centered at $\vec{r} = \vec{b}$

Show that field inside the sphere is: $\vec{E} = \frac{\rho(\vec{r}-\vec{b})}{3\epsilon_0}$

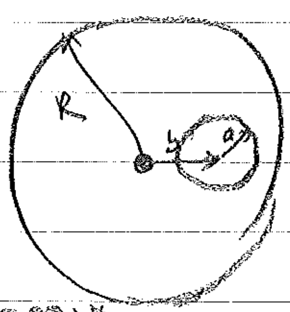


if sphere was centered at origin

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 R^3}\vec{r} = \frac{\rho}{3\epsilon_0}\vec{r}$$

now: $\boxed{\vec{E} = \frac{\rho}{3\epsilon_0}(\vec{r}-\vec{b})}$

b)



$a < b < R$

spherical hole centered at b.

solid sphere has charge density ρ.

$\vec{E}$ inside the hole?

use superposition

$$\vec{E} = \frac{\rho}{3\epsilon_0}\vec{r} - \frac{\rho}{3\epsilon_0}(\vec{r}-\vec{b}) = \boxed{\frac{\rho}{3\epsilon_0}\vec{b}}$$

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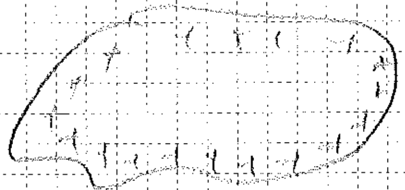
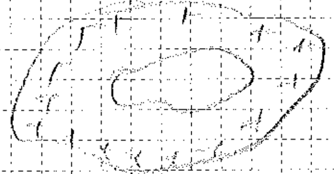
Charges on Conductors:

Consider a cavity inside a conductor.

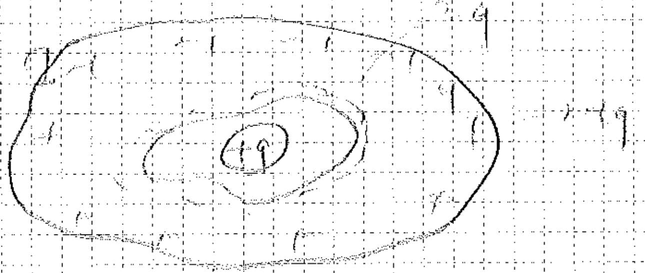




In a conductor all the charge is collected on the outer surface.

In a conductor with a cavity all the charge is collected on the outer surface.

Since E = 0 inside the conductor no charge will be collected on the inner surface according to Gauss' Law.



If there is a charge in the cavity charge will be accumulated on the inner surface such that total charge inside is 0.

+q will be accumulated in the outer surface.

Show fig 22.26, fig 22.29