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— CHAPTER 23 —

Electric Potential Energy

We want to calculate the potential energy function of the electrical force.

Remember: $W_{a\to b} = \int_a^b \vec{F}\cdot d\vec{l}$, work done by force $\vec{F}$ to carry a particle from point a to b

For a conservative force: $\oint_a^a \vec{F}\cdot d\vec{l} = 0$

$$W_{a\to b} = U_a - U_b = -\Delta U,$$ U is the potential energy function

Work-energy theorem states that:

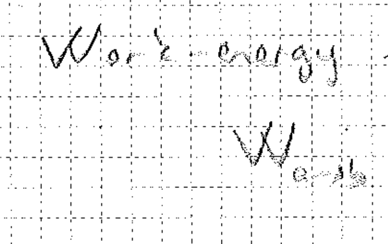
$$W_{a\to b} = \Delta K,$$ work done on an object is equal to the change in it's kinetic energy

if work is done by a conservative force $\vec{F}$,

$$W_{a\to b} = -\Delta U = \Delta K \Rightarrow U_a - U_b = K_b - K_a$$

$$\Rightarrow K_a + U_a = K_b + U_b,$$ Total energy is conserved

Electric Potential Energy in a Uniform Field



Consider a positive charge q_0 in a uniform electric field $\vec{E}$.

$$W_{a\to b} = Fd = q_0 Ed$$, if the motion is in the -y direction

For a general curved motion: $W_{a\to b} = \int_a^b \vec{F}\cdot d\vec{l}$

$\vec{F} = -q_0 E\hat{j}$, $d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k} \Rightarrow W_{a\to b} = \int_{y_a}^{y_b} -q_0 E\, dy = -q_0 E(y_b - y_a)$
$= q_0 Ed$

This is analogous to gravitational force:

$$F_y = -mg \quad , \quad F_y = -q_0 E$$

⇒ We can define electric potential energy similar to the gravitational potential energy:

$$U = mgy \quad , \quad \boxed{U = q_0 E y}$$

$$\Rightarrow W_{a\to b} = -\Delta U = U_a - U_b = q_0 E y_a - q_0 E y_b = q_0 E (y_a - y_b)$$

This equation is also valid when q_0 is negative.

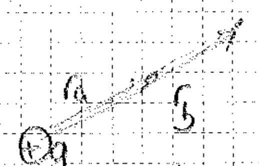
For $q_0 < 0$, if $y_a > y_b$ $\quad W_{a\to b} = \underbrace{q_0}_{<0} E \underbrace{(y_a - y_b)}_{>0} < 0$

if $y_a < y_b$ $\quad W_{a\to b} > 0$

Electric Potential Energy of Two Point Charges:

Consider a stationary point charge q. We want to calculate the work done on a test charge q_0 moving from one point to another under the applied electric force due to charge q.

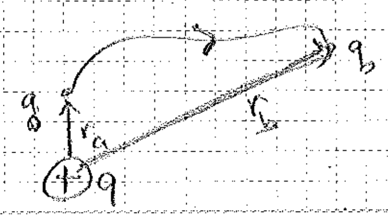
Consider first a radial displacement



$$F_r = \frac{q q_0}{4\pi\epsilon_0}\frac{1}{r^2} \quad \text{in } +\hat{r} \text{ direction.}$$

$$\Rightarrow W_{a\to b} = \int_{r_a}^{r_b} F_r\, dr = \int_{r_a}^{r_b} \frac{1}{4\pi\epsilon_0}\frac{q q_0}{r^2}\, dr = \frac{q q_0}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

Now consider a more general displacement:



$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$$

$$\Rightarrow W_{a\to b} = \int_a^b \vec{F}\cdot d\vec{l}$$

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$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}\hat{r} \Rightarrow W_{a\to b} = \int_{r_a}^{r_b} \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}\,dr = \frac{1}{4\pi\epsilon_0}\, qq_0 \left(-\frac{1}{r}\right)\Big|_{r_a}^{r_b}$$

$$= \frac{1}{4\pi\epsilon_0}\, qq_0\left(\frac{1}{r_a}-\frac{1}{r_b}\right) \quad //$$

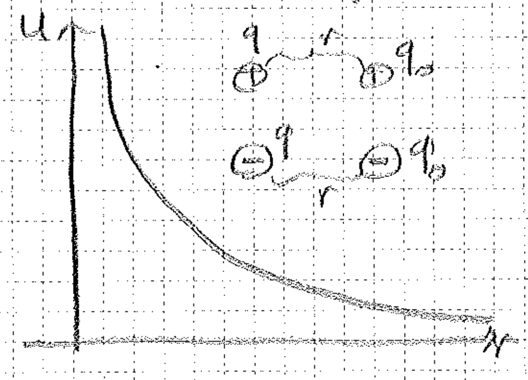
Work done only depends on the initial and final positions but not to the path taken.

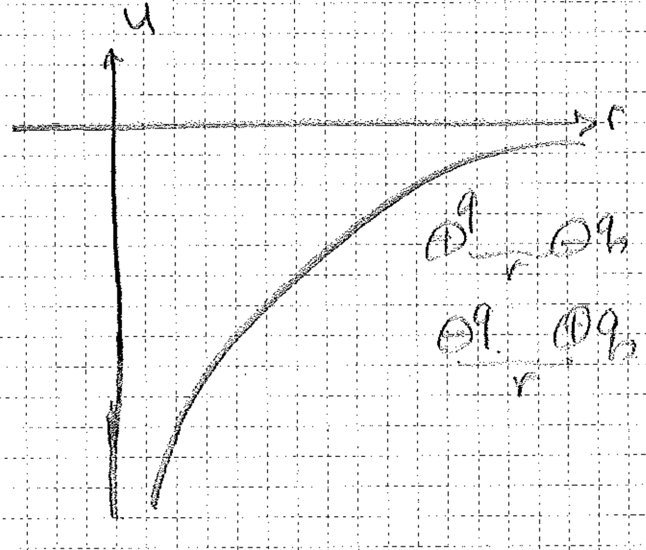
$\therefore$ We can define a potential energy function such that:

$$W_{a\to b} = -\Delta U = U_a - U_b \Rightarrow \boxed{U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}}$$

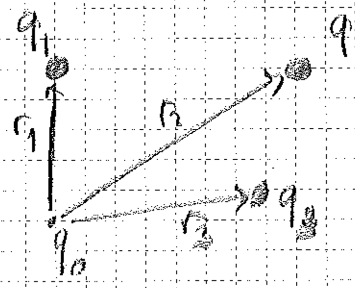
In this definition of potential energy function U is taken to be 0 at $r \to \infty$.

This is the potential energy function due to the interaction between the two charges.





Electric Potential Energy associated with Several Point Charges:



Consider several point charges q_1, q_2, q_3 at distances r_1, r_2, r_3 from a test charge q_0

If the point charge is moved from location a to b:

$$W_{a\to b} = \int_a^b q_0 \vec{E}\cdot d\vec{l} = \underbrace{\int_a^b q_0 \vec{E}_1\cdot d\vec{l}}_{=U_{1a}-U_{1b}} + \int_a^b q_0 \vec{E}_2\cdot d\vec{l} + \int_a^b q_0 \vec{E}_3\cdot d\vec{l}$$

$$= -\Delta U_1 - \Delta U_2 - \Delta U_3$$

$$= \underbrace{(U_{1a} + U_{2a} + U_{3a})}_{U_a} - \underbrace{(U_{1b} + U_{2b} + U_{3b})}_{U_b}$$

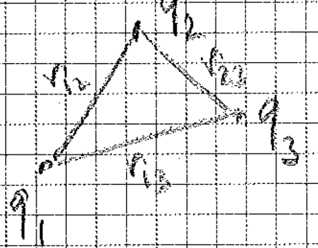
$= -\Delta U$, where U is the electric potential energy associated with the test charge at point a.

∴ In a collection of charges, the potential energy associated with a test charge at point a is:

$$U = \frac{q_0}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \dots\right) = \boxed{\frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}}$$

"algebraic sum of potential energy due to point charges"

This potential energy describes the work that needs to be done in order to bring the charge q_0 from ∞ to its specific position.



The work done by the electric force while forming such a collection of charges is:

$$W = -\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}} - \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r_{13}} - \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{r_{23}}$$

$$= -\Delta U = -(U-0) = -U$$ ← Total potential energy

If there are many point charges:

$$\boxed{U = \frac{1}{4\pi\varepsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}}$$

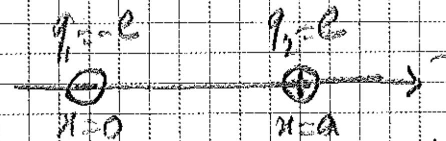
Total potential energy

Sum of the potential energies of interaction for each pair of charges.

$W = -U$; work done by the electric force

$W_{external} = U$; work that needs to be done in order to assemble the charges.

Ex 23.2:



Two point charges q_1 and q_2 are located at $x_1 = 0$ and $x_2 = a$.

a) Work that must be done to bring a charge $q_3 = e$ from infinity to $x = 2a$?

b) Total potential energy of the system of three charges?

a) $$W_{\infty \to 2a} = -\Delta U = U_\infty - U_{2a} = 0 - \left(\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{2a} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{a}\right)$$

$$= -\frac{1}{4\pi\varepsilon_0}\left(-\frac{e^2}{2a} + \frac{e^2}{a}\right) = -\frac{e^2}{8\pi\varepsilon_0 a}$$ ⇐ Work done by the electric force.

Work done by the external force:

$$W_{ext} = -W_{\infty \to 2a} = \frac{e^2}{8\pi\varepsilon_0 a}$$ //

b) $$U_{tot} = \frac{1}{4\pi\epsilon_0} \sum_{i<j} \frac{q_i q_j}{r_{ij}} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

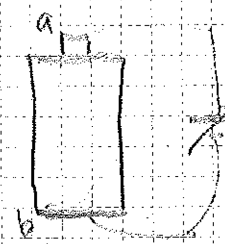
$$= \frac{1}{4\pi\epsilon_0}\left(\cancel{\frac{-e^2}{a}} + \frac{-e^2}{2a} + \cancel{\frac{e^2}{a}}\right) = \boxed{-\frac{e^2}{8\pi\epsilon_0 a}}$$

Electric Potential

Electric potential is potential energy associated with a test charge q_0 per unit charge.

$$V = \frac{U}{q_0} \quad \longleftrightarrow \text{Analogous to } \quad \vec{E} = \frac{\vec{F}}{q_0}$$

units (Volt) = $\frac{(\text{Joule})}{(\text{Coulomb})}$, $1V = 1 J/C$



In a battery

$V_{ab} = 1.5V$

$$W_{a\to b} = -\Delta U$$

$$\Rightarrow \frac{W_{a\to b}}{q_0} = -\frac{\Delta U}{q_0} = \frac{U_a}{q_0} - \frac{U_b}{q_0} = V_a - V_b$$

Work done by the electric force is $W_{a\to b} = (V_a - V_b)q_0$

$V_{ab} = V_a - V_b$ is the work done per unit charge by the electric force when a charged body moves from a to b.

→ Electric potential due to a point charge:

$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

→ Electric potential due to a collection of point charges:

$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$

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Electric potential due to a continuous distribution of charge:

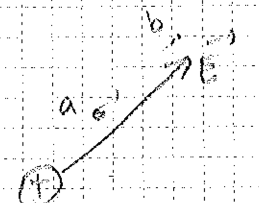
$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}$$

It is also possible to work backwards and find the electric potential from the electric field:

$$W_{a\to b} = \int_a^b \vec{F}\cdot d\vec{l} = \int_a^b q_0\vec{E}\cdot d\vec{l} \Rightarrow \boxed{V_a - V_b = \int_a^b \vec{E}\cdot d\vec{l}}$$

Independent from the path taken from a to b.

$$\text{or} \quad V_a - V_b = -\int_b^a \vec{E}\cdot d\vec{l}$$



$V_a - V_b > 0$ if you move in the direction of $\vec{E}$.

$\Rightarrow V_A > V_b$

so Electric potential decreases if you move in the direction of $\vec{E}$

$V_a - V_b < 0$ if you move in the direction opposite to $\vec{E}$.

$\Rightarrow$ Electric potential increases if you move in the direction opposite to $\vec{E}$.

Ex 23.3: A proton moves in a straight line from a to b, a total distance $d = 0.5m$. Electric field is uniform, $E = 1.5\times10^{7}\ V/m = 1.5\times10^{7}\ N/C$

a) The force on proton? $\Rightarrow qE = (1.6\times10^{-19})\times1.5\times10^{7}\ N/C = 2.4\times10^{-12}\ N$

b) Work done by the field? $\rightarrow dqE = 1.2\times10^{-12}\ J$, it is +

c) Potential difference $V_a - V_b$? $\Rightarrow V_a - V_b = \frac{W_{a\to b}}{q} = \frac{1.2\times10^{-12}\ J}{1.602\times10^{-19}\ C}$

$$= 7.5\times10^{6}\ J/C$$

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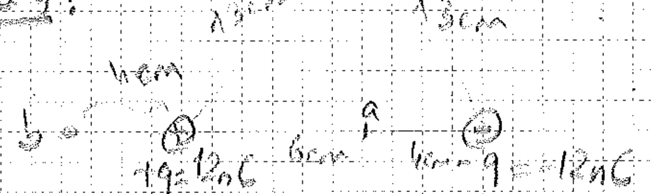
Ex 23.4: Potentials at points a, b, and c.



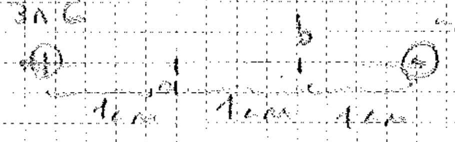
Use $V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$

At point a: $V_a = \frac{1}{4\pi\epsilon_0}\left(\frac{12\times10^{-9}C}{6\times10^{-2}m} - \frac{12\times10^{-9}C}{4\times10^{-2}m}\right) = -900V$

At point b: $V_b = \frac{1}{4\pi\epsilon_0}\left(\frac{12\times10^{-9}C}{4\times10^{-2}m} - \frac{12\times10^{-9}C}{14\times10^{-2}m}\right) = 1930V$

At point c: $V_c = 0$

Ex 23.7: A particle with mass $m = 5\times10^{-9}$ kg and charge $q_0 = 2nC$ starts from rest and moves from a to b. What is its speed at b?



Use the conservation of energy.

$U_a + K_a = U_b + K_b$

$K_a = 0$

$U_a = q_0 V_a = q_0\left(\frac{1}{4\pi\epsilon_0}\left(\frac{3nC}{1cm} - \frac{3nC}{2cm}\right)\right)$

$U_b = q_0 V_b$

$\Rightarrow q_0 V_a = q_0 V_b + \frac{1}{2} m v^2 \Rightarrow v = \sqrt{\frac{2 q_0 (V_a - V_b)}{m}}$

$V_a - V_b = \frac{3\times10^{-9}C}{4\pi\epsilon_0}\left\{\frac{1}{10^{-2}m} - \frac{1}{2\times10^{-2}m} - \left(\frac{1}{2\times10^{-2}m} - \frac{1}{10^{-2}m}\right)\right\}$

$= \frac{3\times10^{-9}C}{4\pi\epsilon_0}\left(\frac{1}{10^{-2}m}\right) = 9\times10^{9}\frac{Nm^2}{C^2}\times\frac{3\times10^{-9}C}{10^{-2}m} = 2700V$

$\Rightarrow \boxed{v = 46\, m/s}$

Calculating Electric Potential :

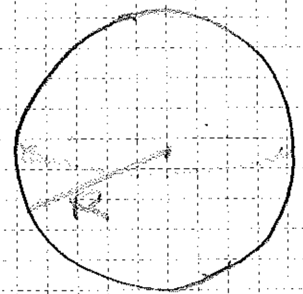
We will use :

$$V = \underbrace{\frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}}_{\text{If the charge distribution is known}} \quad \text{or} \quad \underbrace{V_a - V_b = \int_a^b \vec{E}\cdot d\vec{l}}_{\text{If the electric field is known}}$$

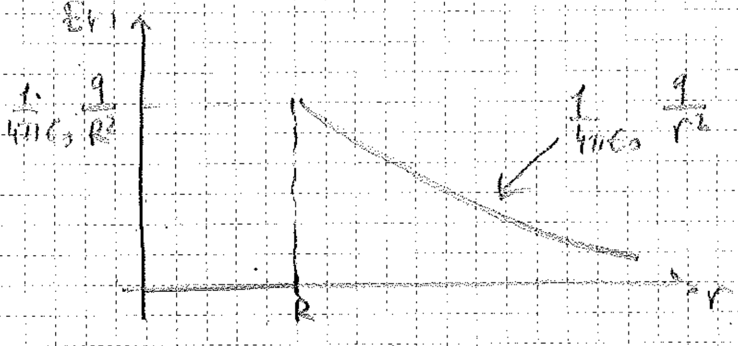
Examples:

Ex 23.8: A charged conducting sphere with a total charge q.

Find the potential everywhere, both inside and outside.



We know the electric field distribution:



Take $V=0$ at $r \to \infty$ as the reference potential.

$$\Rightarrow V(r_0) - V(\infty) = \int_{r_0}^{\infty} \vec{E}\cdot d\vec{l} \Rightarrow V(r_0) = \int_{r_0}^{\infty} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr = \frac{q}{4\pi\epsilon_0}\left.-\frac{1}{r}\right|_{r_0}^{\infty}$$

$$= \frac{q}{4\pi\epsilon_0}\left(0 + \frac{1}{r_0}\right) = \boxed{\frac{q}{4\pi\epsilon_0 r_0}}$$

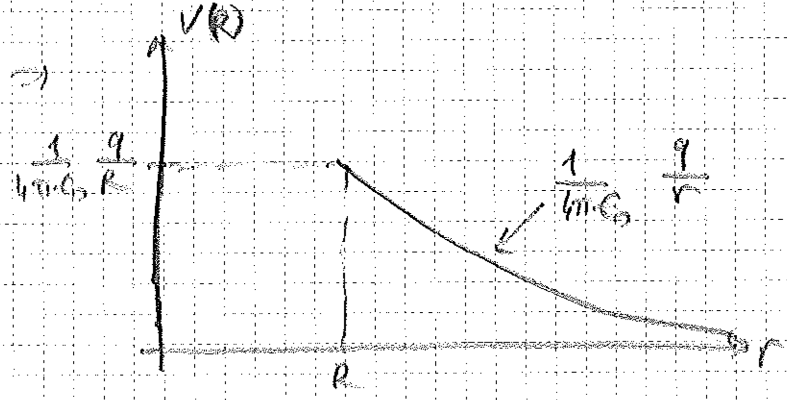
Outside the sphere

$V(R) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$

inside the sphere

$$V(r) - V(R) = \int_r^R \underbrace{\vec{E}}_{0} \cdot d\vec{l} = 0 \Rightarrow \boxed{V(r) = V(R)}$$

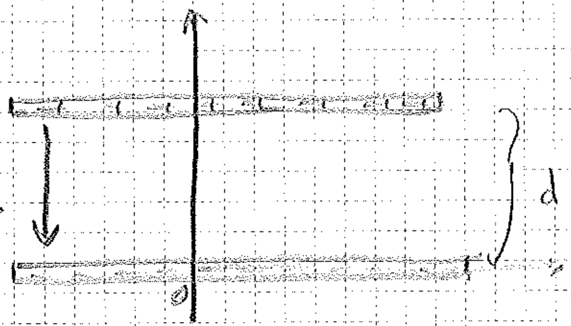
$\Rightarrow$



Ex 23.9: Oppositely Charged Parallel Plates

Potential at any height between two oppositely charged parallel plates:

$\vec{E}$; uniform electric field.



$\vec{E} = -E\hat{j}$, $U = q_0 E y$, potential energy

choose $U = 0$, $V = 0$ at $y = 0$,

$$\Rightarrow V(y) - V(0) = \int_y^0 \vec{E} \cdot d\vec{l} \Rightarrow V(y) = \int_y^0 -E\,dy = -E(0 - y) = Ey$$

$$\Rightarrow \boxed{V(y) = Ey}$$

In general if $V(0) \neq 0$, $V(0) = V_b$

$$\Rightarrow \boxed{V(y) = V_b + Ey}$$

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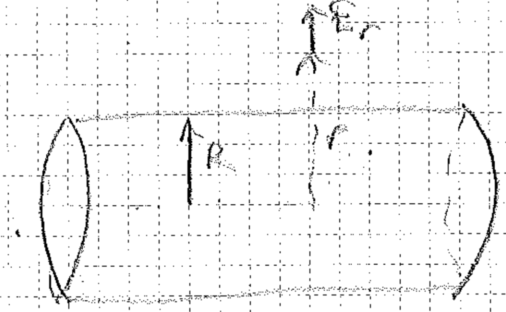
Ex 23.10: An infinite line charge or charged conducting cylinder



Potential a distance r from a very long line of charge with linear charge density λ.

$E(r)\, 2\pi r l = \frac{\lambda l}{\epsilon_0} \Rightarrow E(r) = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$, from Gauss's law.

Take $V(r) = 0$ at $r \to \infty$

$$\Rightarrow V(r_a) - V(\infty) = \int_{r_a}^{\infty} \vec{E}\cdot d\vec{l} = V(r_a) = \int_{r_a}^{\infty} \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\,dr$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln(r)\Big|_{r_a}^{\infty} = \infty$$

$\Rightarrow$ choice of $V(r) = 0$ at $r \to \infty$ is not a good choice.

choose $V(r) = 0$ at a specific r_0 value.

$$\Rightarrow V(r_a) - V(r_0) = \int_{r_a}^{r_0} \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\,dr = \boxed{\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_0}{r_a}\right) = V(r_a)}$$

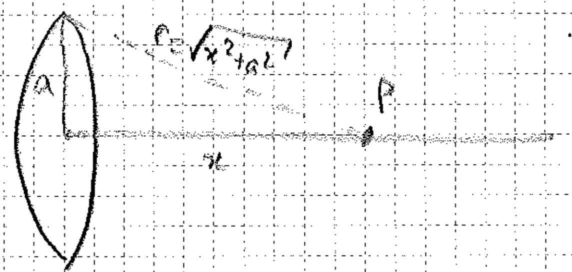
This is also valid for the points outside the cylinder.
For this case we can choose $V(R) = 0$, R: radius of the cylinder.

$$\Rightarrow \boxed{V(r_a) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{R}{r_a}\right)}$$

Ex 23.11 : A ring of charge

Electric charge is uniformly distributed around a thin ring of radius a, with a total charge Q.

Potential at a point on the ring axis a distance x away from the center of the ring.



use: $V = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{r}$

Consider an infinitesimal element of charge dq:

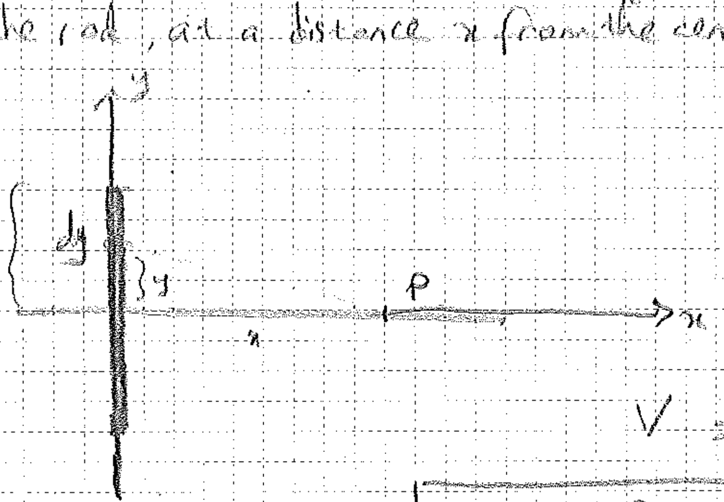
$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{x^2+a^2}} \Rightarrow V = \int dV = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2+a^2}} \int dq$$

$$= \boxed{\frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2+a^2}}}$$

Ex 23.12 : A line of charge

Electric charge Q is distributed uniformly along a line of length 2a.

Potential at a point P along the perpendicular bisector of the rod, at a distance x from the center?



$dq = \lambda dy$

$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{x^2+y^2}}$$

$$\Rightarrow V = \int_{-a}^{a} \frac{1}{4\pi\epsilon_0} \lambda \frac{dy}{\sqrt{x^2+y^2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}\right)$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \frac{Q}{2a} \ln\left(\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}\right)}$$

Equipotential Surfaces:

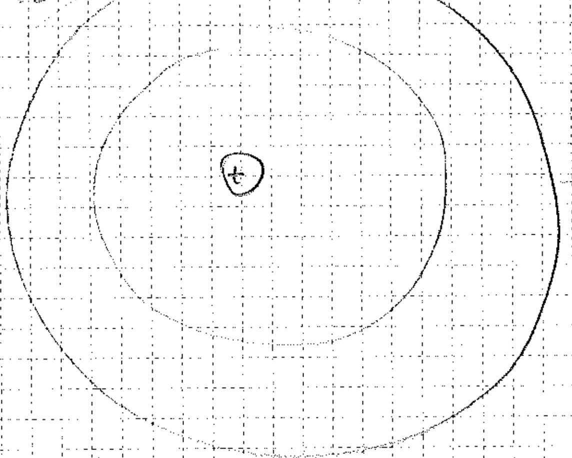
An equipotential surface is a 3D surface on which the electric potential V is the same at every point.

Potential energy doesnot change as a test charge moves over an equipotential surface => electric field cando nowork on such a charge.

$$W_{a\to b} = \int_a^b q_0 \vec{E}\cdot d\vec{l} = q_0 (V_a - V_b) = 0$$

$\Rightarrow \vec{E}\cdot d\vec{l} = 0$ on the equipotential surface

$\Rightarrow \vec{E}$ and equipotential surfaces are always mutually orthogonal.

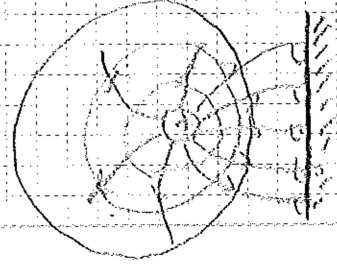


→ spherical equipotential surface of a point charge.

Equipotentials and Conductors:

When all charges are at rest, the surface of a conductor is always an equipotential surface.

⇒ When all charges are at rest, the electric field just outside a conductor must be perpendicular to the surface at every point.



Potential Gradient:

Relationship $V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l}$, can also be used to calculate the electric field given the potential function.

$$V_a - V_b = -\int_a^b dV = \int_a^b \vec{E} \cdot d\vec{l}$$, this is true for any pair of points a, b.

$$\Rightarrow -dV = \vec{E} \cdot d\vec{l}$$

$$\vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k}, \quad d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\Rightarrow -dV = E_x dx + E_y dy + E_z dz$$

$\Rightarrow$ Choose dl such that $dy = dz = 0$

$$\Rightarrow -dV = E_x dx \Rightarrow E_x = -\left(\frac{dV}{dx}\right)_{y,z\ \text{constant}} \equiv \underbrace{-\frac{\partial V}{\partial x}}_{\text{partial derivative}}$$

similarly $E_y = -\frac{\partial V}{\partial y}$, $E_z = -\frac{\partial V}{\partial z}$

$$\Rightarrow \vec{E} = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

$$= -\vec{\nabla} V \quad \text{where} \quad \vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right), \text{ gradient operator}$$

In spherical coordinates, cylindrical coordinates

$$\boxed{E_r = -\frac{\partial V}{\partial r}}$$

radial component of the E-field

Ex 23.13

Potential of a point charge $V = \frac{q}{4\pi\epsilon_0 r}$

$\Rightarrow$ Electric field; $E_r = -\frac{-q}{4\pi\epsilon_0 r^2} = \frac{q}{4\pi\epsilon_0 r^2}$ //

Ex 23.14 :

Potential outside a conducting cylinder

$V = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{R}{r} \Rightarrow$

Electric field ; $E_r = -\frac{dV}{dr} = -\frac{\lambda}{2\pi\epsilon_0} \frac{d}{dr}\left(\ln \frac{R}{r}\right)$

$= +\frac{\lambda}{2\pi\epsilon_0} \frac{d}{dr}(\ln r) = \frac{\lambda}{2\pi\epsilon_0 r}$ //

Ex 23.15 : Potential of a ring of charge

For a ring of charges $V = \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{x^2+a^2}}$

$\Rightarrow$ Electric field;

$E_x = -\frac{dV}{dx} = -\frac{1}{4\pi\epsilon_0} Q \frac{2x \cdot \left(-\frac{1}{2}\right)}{(x^2+a^2)^{3/2}} = \boxed{\frac{1}{4\pi\epsilon_0} \frac{Qx}{(x^2+a^2)^{3/2}}}$ //