

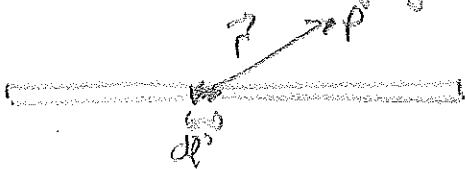
# Chapter 28: Sources of Magnetic Field

How is a magnetic field generated?

Stationary charges generate Electric Field (Coulomb's Law)

Moving charges (current) generate Magnetic Field (Biot-Savart Law)

Consider a current carrying wire



The magnetic field generated by a piece with length  $dl$  is:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$$

Biot-Savart Law

Compare with Coulomb's Law

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \hat{r}$$

$d\vec{l}$ : vector with length  $dl$  in the same direction as the current flow

$\hat{r}$ : vector from the point on the wire to the observation point  $P$ .  $\hat{r}$  unit vector in the direction of  $\vec{r}$ .  $\hat{r} = \frac{\vec{r}}{r}$

$P$ : observation point

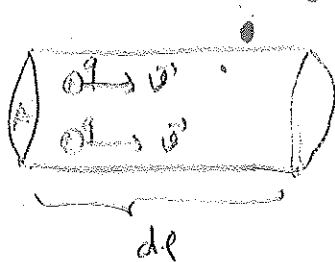
$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times \hat{r}}{r^2}$$

integrated over the current carrying wire

$\mu_0$ : magnetic permeability constant

Examples: Magnetic field due to a moving charge  $q$

Consider the individual charges making up the current flow



$$I = nq\vec{v}A$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{nq\vec{v}A d\vec{l} \times \hat{r}}{r^2}$$

$\vec{v}$  is in the same direction as current  $d\vec{l}$

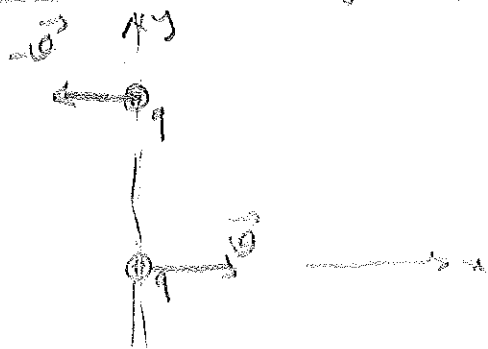
$$\rightarrow d\vec{B} = \frac{\mu_0}{4\pi} \frac{nq dA \vec{v} \times \hat{r}}{r^2}$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{Nq \vec{v} \times \hat{r}}{r^2} \rightarrow \text{number of charges}$$

∴ Field due to a single charge

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q \vec{v} \times \hat{r}}{r^2}$$

Ex 28.1: Two moving charges



Two protons move parallel to x-axis in opposite directions. Electric and magnetic forces on the upper proton and the ratio of their magnitudes?

$$\vec{F}_e = q\vec{E} = q \cdot \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} = \frac{q^2}{4\pi\epsilon_0 r^2} \hat{r}$$

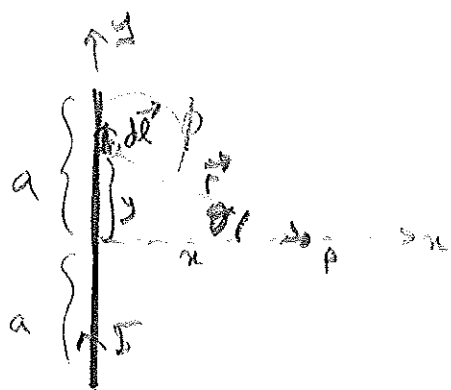
$$\vec{F}_B = q\vec{v} \times \vec{B} = q \left( -v\hat{i} \times \left( \frac{\mu_0}{4\pi} \frac{q}{r^2} \vec{v} \times \hat{r} \right) \right) = -qv \left( \hat{i} \times \left( \frac{\mu_0 q}{4\pi} \frac{v}{r^2} \hat{z} \times \hat{j} \right) \right)$$

$$= -\frac{q^2 \mu_0 v^2}{4\pi r^2} (\hat{z} \times \hat{k}) = -\frac{\mu_0 q^2 v^2}{4\pi r^2} (-\hat{j}) = \frac{\mu_0 q^2 v^2}{4\pi r^2} \hat{j}$$

$$\Rightarrow \frac{F_B}{F_e} = \frac{\mu_0 q^2 v^2}{4\pi r^2} \frac{4\pi\epsilon_0 r^2}{q^2} = \mu_0 \epsilon_0 v^2 = \frac{1}{c^2} v^2 = \boxed{\frac{v^2}{c^2}}$$

∴ when  $v \ll c$   $F_B \ll F_e$ !

# Magnetic Field of a Straight Current-Carrying Conductor



$\vec{B}$  at a point a distance  $x$  from the conductor on its perpendicular bisector.

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$$

$$d\vec{l} \times \hat{r} = -dl \hat{k} \sin\theta \Rightarrow d\vec{B} = -\frac{\mu_0}{4\pi} I \frac{dl}{(x^2 + y^2)} \hat{k} \left(\frac{x}{r}\right), \quad dl = dy$$

$$\Rightarrow B = \int dB = \frac{\mu_0 I}{4\pi} \int \frac{dy \cdot x}{r^3} \quad \begin{matrix} \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{y}{x} = \tan\theta \\ \theta_{max} \quad \Rightarrow \quad dy = \frac{x}{\cos^2\theta} d\theta \end{matrix}$$

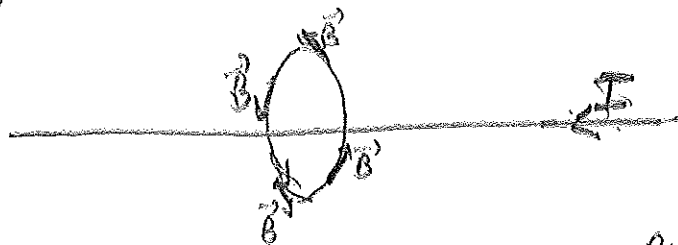
$$= \frac{\mu_0 I}{4\pi} \int \frac{x}{\cos^2\theta} d\theta \cdot \frac{x}{x^3} \cos^3\theta = \frac{\mu_0 I}{4\pi x} \int \cos\theta d\theta$$

$$= \frac{\mu_0 I}{4\pi x} 2 \sin\theta_{max} = \frac{\mu_0 I}{2\pi x} \cdot \frac{a}{\sqrt{a^2 + x^2}}$$

$$B = \frac{\mu_0 I}{2\pi} \frac{a}{x \sqrt{a^2 + x^2}}$$

for  $a \rightarrow \infty$   $\lim_{a \rightarrow \infty} B = \frac{\mu_0 I}{2\pi x}$

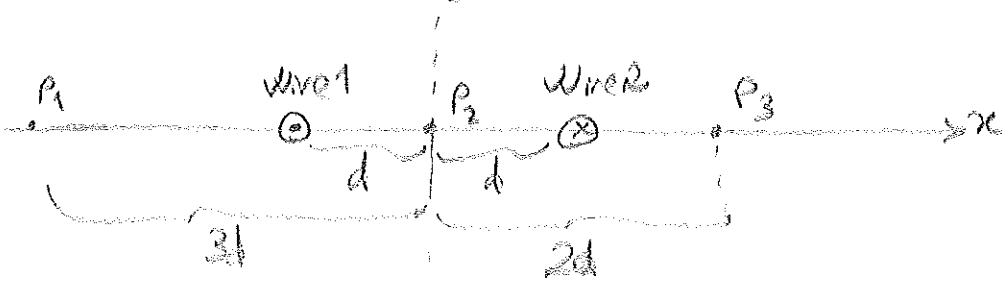
Problem is symmetric for rotations around  $y$ -axis.  $\vec{B}$  has the same magnitude at all points on a circle centered on the conductor.  $\vec{B}$  is in the tangential direction to this circle.



$$B = \frac{\mu_0 I}{2\pi r}, \text{ long straight, current-carrying conductor}$$

Right hand rule gives the direction.

### Example 28.4: Magnetic field of two wires



Each wire carries a current  $I$ .

a) Magnitude and direction of  $\vec{B}$  at  $P_1, P_2, P_3$ ?

APPLY SUPERPOSITION

$$\text{at } P_1: \vec{B}_1 = -\frac{\mu_0 I}{2\pi \cdot 2d} \hat{j} + \frac{\mu_0 I}{2\pi \cdot 4d} \hat{j} = -\frac{\mu_0 I}{2\pi \cdot 4d} \hat{j} //$$

$$\vec{B}_2 = \frac{\mu_0 I}{2\pi \cdot 2d} \hat{j} + \frac{\mu_0 I}{2\pi \cdot 2d} \hat{j} = \frac{\mu_0 I}{\pi d} \hat{j} //$$

$$\vec{B}_3 = \frac{\mu_0 I}{2\pi \cdot 3d} \hat{j} - \frac{\mu_0 I}{2\pi \cdot d} \hat{j} = -\frac{2\mu_0 I}{2\pi \cdot 3d} \hat{j} //$$

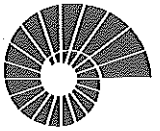
b) Magnitude and direction of  $\vec{B}$  at any points to the right of wire 2?

$$\vec{B}(x) = -\frac{\mu_0 I}{2\pi (x-d)} \hat{j} + \frac{\mu_0 I}{2\pi (x+d)} \hat{j}$$

$$= -\frac{\mu_0 I}{2\pi} \hat{j} \left\{ \frac{1}{x-d} - \frac{1}{x+d} \right\} = -\frac{\mu_0 I d}{\pi (x^2 - d^2)} \hat{j}$$

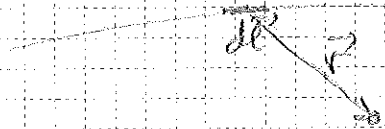
for  $x \rightarrow \infty$

$$\boxed{\lim_{x \rightarrow \infty} \vec{B}_a = -\frac{\mu_0 I d}{\pi x^2} \hat{j} //$$

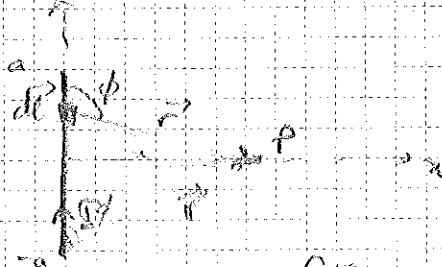


Summary: 
$$dB = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$$

Biot and Savart Law



Magnetic Field of a Straight Current carrying wire:



$$d\vec{l} = dl \hat{j}, \quad \hat{r} = \cos\phi \hat{i} + \sin\phi \hat{j}$$

$$d\vec{l} \times \hat{r} = -\sin\phi \hat{k} dl$$

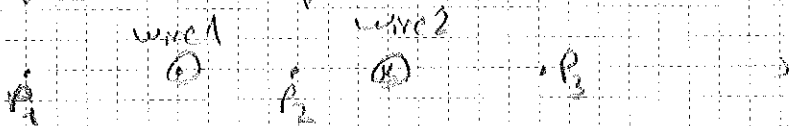
$$\vec{B} = \int dB = \frac{\mu_0 I}{4\pi} \int \frac{-\sin\phi dl \hat{k}}{r^2}$$

$$= -\frac{\mu_0 I}{4\pi} \int \frac{x}{r^3} dy \hat{k} = -\frac{\mu_0 I}{4\pi} \int_{-a}^a \frac{x}{(x^2 + y^2)^{3/2}} dy \hat{k}$$

$$= -\frac{\mu_0 I}{2\pi} \frac{a}{x\sqrt{a^2 + x^2}} \hat{k}$$

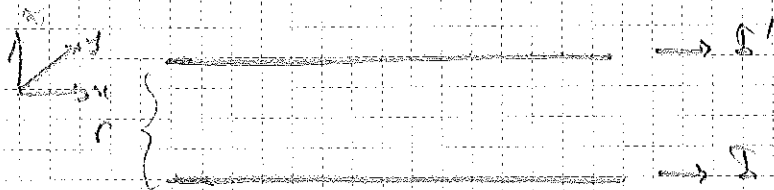
Mag 
$$\vec{B} = \frac{\mu_0 I}{2\pi x} \hat{k}$$
, given by the right hand rule.

Magnetic Field of two wires:



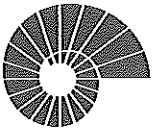
Vector addition of  $\vec{B}$  due to each wire.

Force between Parallel Conductors:



$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{j}$$

Force on the upper conductor,  
$$\vec{F} = I \vec{L} \times \vec{B}$$



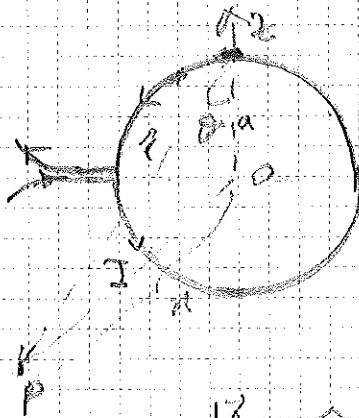
$$\vec{F} = I' L \hat{i} \times \frac{\mu_0 I}{2\pi r} (-\hat{j}) = \left( \frac{\mu_0 I I'}{2\pi r} L \right) - \hat{k}$$

Force per unit length:  $\left[ \frac{\vec{F}}{L} = - \frac{\mu_0 I I'}{2\pi r} \hat{k} \right]$  attractive!

For the other wire:

$$\frac{\vec{F}}{L} = \frac{\mu_0 I I'}{2\pi r} \hat{k} \quad \text{: attractive, due to symmetry}$$

Magnetic Field of a Circular Current Loop:



Magnetic field at a point on the axis of the loop.

$$dB = \frac{\mu_0 I}{4\pi} \frac{dl \times \hat{r}}{r^2}$$

$dl$  is orthogonal to  $\hat{r}$

$$dl \times \hat{r} = dl \hat{\phi}$$

$$dl = -\hat{j} dl$$

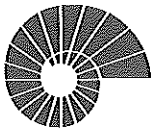
$$\hat{r} = -\cos\theta \hat{k} + \sin\theta \hat{i}$$

$$\rightarrow dl \times \hat{r} = \cos\theta dl \hat{i} + \sin\theta dl \hat{k}$$

$$B_x = \int dB_x = \int \frac{\mu_0 I}{4\pi} \frac{\cos\theta dl}{r^2} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a}{(a^2+x^2)^{3/2}} a d\phi = \frac{\mu_0 I a^2}{2(a^2+x^2)^{3/2}}$$

$B_z = 0$  due to rotational symmetry

$$B_x = \frac{\mu_0 m}{2\pi (a^2+x^2)^{3/2}} \quad \mu = I\pi a^2 \in \text{magnetic dipole moment}$$



$$\begin{aligned} d\vec{l} &= dx\hat{i} + dy\hat{j} + dz\hat{k} \\ d\vec{r} &= dr\hat{r} + r d\theta\hat{\theta} \\ d\vec{r} &= dr\hat{r} + r d\theta\hat{\theta} + r \sin\theta d\phi\hat{\phi} \\ d\vec{l} &= r d\phi\hat{\phi} + r d\theta\hat{\theta} + dr\hat{r} \end{aligned}$$

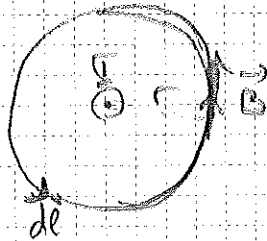
Ampère's Law

Similar to the Gauss' Law  $\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$ , there is a simple relationship between  $\vec{B}$  and  $I$ .

The relationship now involves:

$\oint \vec{B} \cdot d\vec{l}$ ; the integral of  $\vec{B}$  around a closed path.

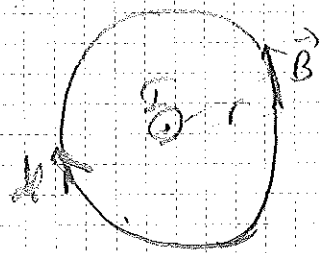
Consider the magnetic field caused by a long, straight, conductor carrying a current  $I$ :



$B = \frac{\mu_0 I}{2\pi r}$ , in the tangential direction.

$\oint \vec{B} \cdot d\vec{l}$  circle with radius  $r$  integration path counter clockwise

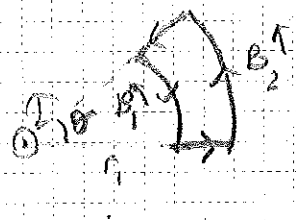
$$\int B \hat{\phi} \cdot dl \hat{\phi} = B \int dl = 2\pi r B = \mu_0 I$$



clockwise direction

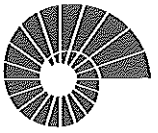
$$\begin{aligned} \oint \vec{B} \cdot d\vec{l} &= \int B \hat{\phi} \cdot dl (-\hat{\phi}) \\ &= -B \int dl = -B 2\pi r = -\mu_0 I \end{aligned}$$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$  current which obeys the right-hand rule.  
closed path integral around a circle.

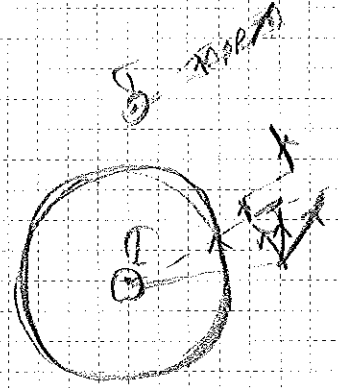
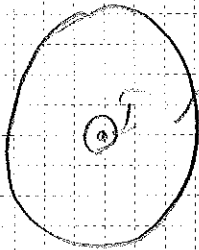


Now consider a closed path that does not enclose the current.

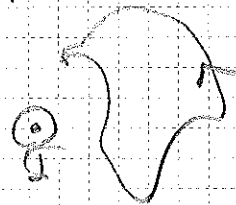
$$\begin{aligned} \oint \vec{B} \cdot d\vec{l} &= \int_{\text{top}} B_1 dl + \int_{\text{right}} B_2 dl \\ &= -B_1 \int dl + B_2 \int dl = -B_1 \frac{2\pi r_1}{2\pi} + B_2 \frac{2\pi r_2}{2\pi} \end{aligned}$$



$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = -\frac{\mu_0 I}{2\pi r} \int_0^{\theta} r d\theta + \frac{\mu_0 I}{2\pi r} \int_{\theta}^{\theta} r d\theta = 0 //$$



These results considering the special paths can be generalized

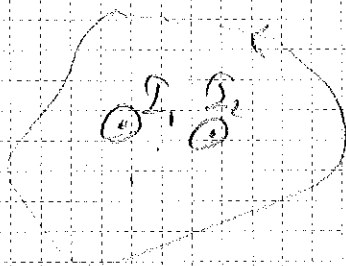


$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$\oint \vec{B} \cdot d\vec{l} = 0 = \mu_0 I \uparrow$$

enclosed current

If there are more than one current carrying wires:



Superposition:

$$\mu_0 I_1 + \mu_0 I_2 + \dots = \oint \vec{B}_1 \cdot d\vec{l} + \oint \vec{B}_2 \cdot d\vec{l} + \dots$$

$$\Rightarrow \mu_0 I_{\text{encl}} = \oint \vec{B} \cdot d\vec{l}$$

$$I_{\text{encl}} = I_1 + I_2 + \dots$$

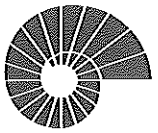
$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \dots$$

$\therefore$  Ampère's Law:

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{encl}}}$$

algebraic sum of currents enclosed by the integration path.



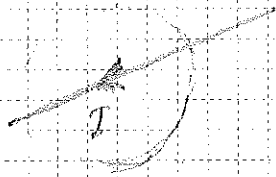


Applications of Ampère's Law:

We will benefit from the symmetry of the problems and use the Ampère's Law to determine the  $\vec{B}$  field.

Ex: 28.8

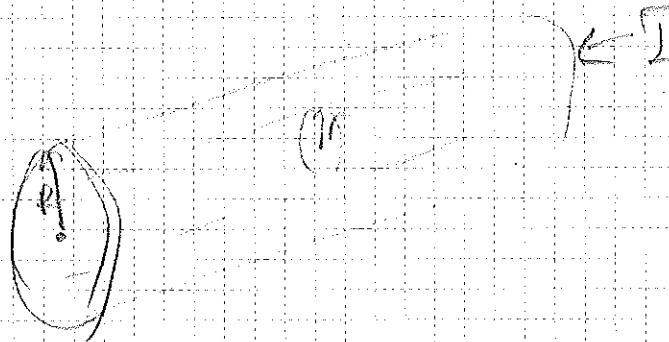
Field of a long, straight, current-carrying wire:



Due to symmetry  $\vec{B}$  with a distance  $r$  from the wire should have the same magnitude and tangential direction.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \quad \Rightarrow \quad B \cdot 2\pi r = \mu_0 I \quad \Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

Ex 28.9: Field inside a long cylindrical conductor



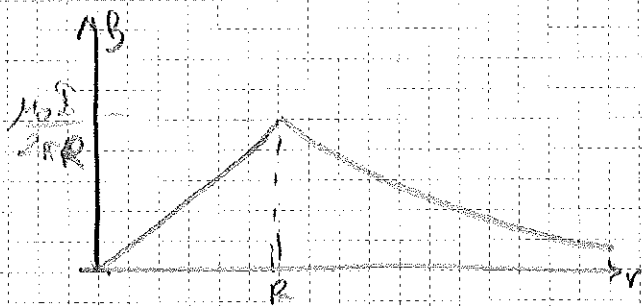
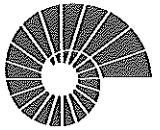
Consider a long cylindrical conductor with a radius  $R$ , carrying a current  $I$ .

The field at a distance  $r < R$  from the center:

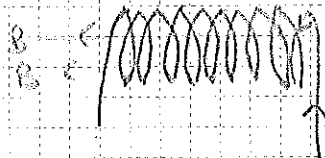
$$\oint \vec{B} \cdot d\vec{l} = B \cdot 2\pi r = \mu_0 \frac{I}{\pi R^2} \pi r^2 \quad \Rightarrow \quad B = \frac{\mu_0 I}{2\pi} \frac{r}{R^2} = \frac{\mu_0 I}{2\pi R^2} r$$

At a distance  $r > R$  from the center

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I = B \cdot 2\pi r \quad \Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

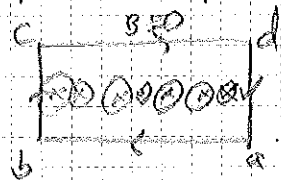


Ex. 28.12: Field of a solenoid:



Field lines towards the center of a solenoid are nearly parallel.  
 External fields are almost equal to zero.

→ Choose an amperian loop:

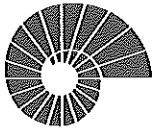


$$\int \vec{B} \cdot d\vec{l} = B L = \mu_0 I N$$

number of loops enclosed

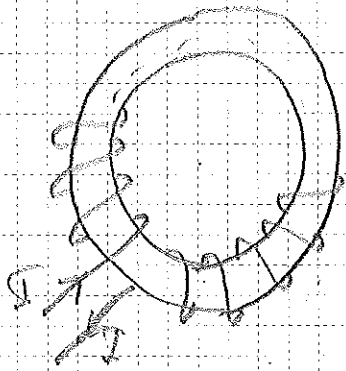
if  $n = \frac{N}{L}$ , number of turns per unit length

$$\boxed{B = \mu_0 I n}$$

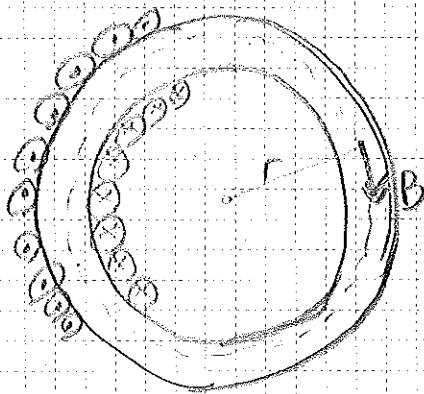


Ex. 28.11.

Field of a toroidal solenoid.



Consider a toroidal solenoid having a total of  $N$  turns.

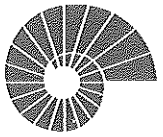


If the coils are very tightly wound, we can consider circular loops in the solenoid.

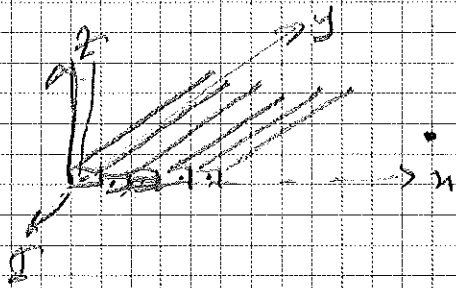
$\Rightarrow B$  is constant and tangential in direction.

Consider a circular ampere loop.

$$B(2\pi r) = \mu_0 I N \Rightarrow \boxed{B = \frac{\mu_0 I N}{2\pi r}} \text{ in the tangential direction.}$$



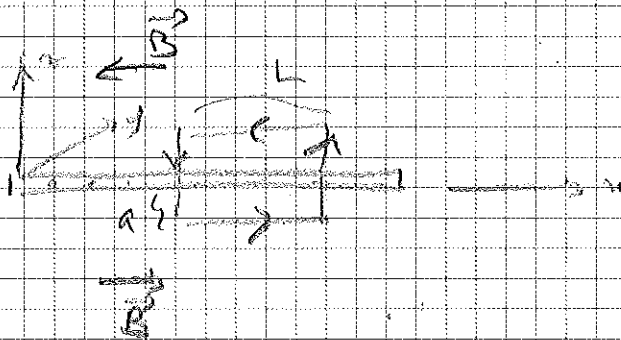
Prob 28.29: Infinite current sheet.



Long, straight conductors with square cross-sections each carrying current  $I$  are laid side-by-side to form an infinite current sheet.

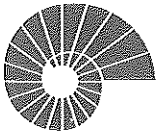
There are  $n$  conductors per unit length along the  $x$ -axis.

Magnitude and direction of the magnetic field at a distance  $a$  below and above the current sheet?

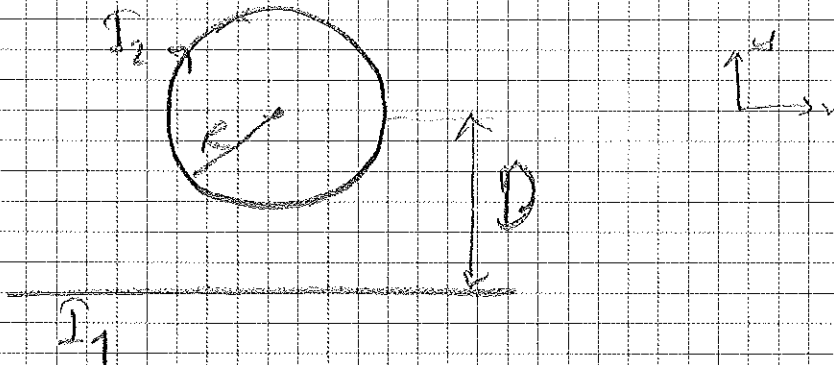


Due to symmetry

$$\oint \vec{B} \cdot d\vec{l} = 2BL = \mu_0 I n L \Rightarrow \boxed{B = \frac{\mu_0 I n}{2}} \text{ constant}$$



Prob 28.24:



What are the magnitude and direction of  $\vec{I}_1$  if  $\vec{B} = 0$  at the center of the circular loop?

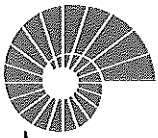
$$dB = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2} \Rightarrow \vec{B}_2 = \frac{\mu_0 I_2}{4\pi} \frac{1}{R^2} 2\pi R (-\hat{y})$$
$$= \frac{\mu_0 I_2}{2} \frac{-\hat{y}}{R}$$

$I_1$  should flow down in direction.

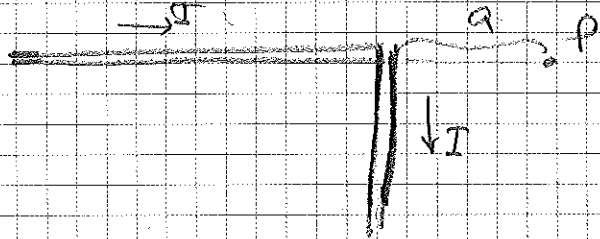
$$\Rightarrow \vec{B}_1 = \frac{\mu_0 I_1}{2\pi D} \hat{y}$$

$$B_1 = B_2 \Rightarrow \frac{\mu_0 I_1}{2} \frac{1}{R} = \frac{\mu_0 I_2}{2\pi D} \Rightarrow$$

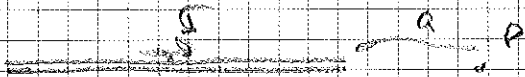
$$\boxed{I_1 = \frac{\pi D}{R} I_2}$$



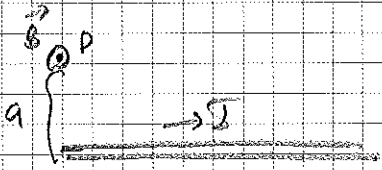
Prob:  
28.68



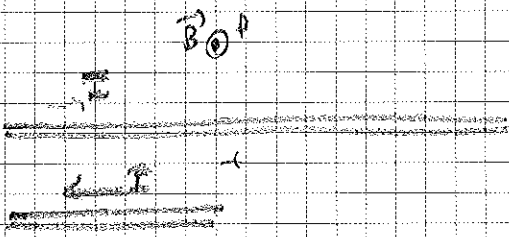
The wire is infinitely long and carries a current  $I$ .  
What is  $\vec{B}$  at point P?



$\vec{B} = 0$



$\vec{B} = ?$



$$B = \frac{\mu_0 I}{2\pi a} \quad \vec{B} \rightarrow \quad \left| B = \frac{\mu_0 I}{4\pi a} \right|$$

