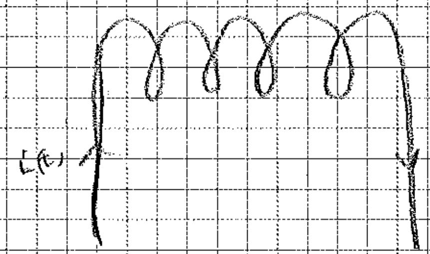
Chapter 30: Inductance

Consider a copper wire wrapped to form a coil



Changing current in the coil will induce an emf in the coil itself.

Such a coil is called an inductor and the relationship between the current and emf reveals the inductance.

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

Total induced emf

Faraday's Law

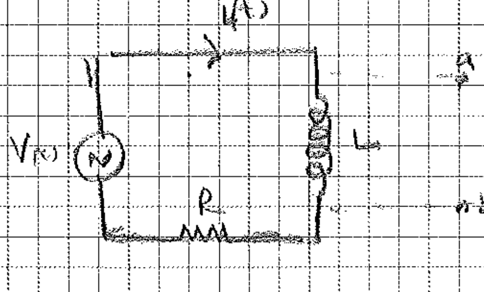
$i(t) = \frac{N}{L}\Phi_B(t) \Rightarrow \boxed{\frac{d\Phi_B}{dt} = \frac{L}{N}\frac{di}{dt}}$

we define, $L = \frac{N\Phi_B}{i}$, as the self-inductance

→ flux flowing through one winding

with this definition, Faraday's Law can be written as:

$$\boxed{\mathcal{E} = -L\frac{di}{dt}}$$

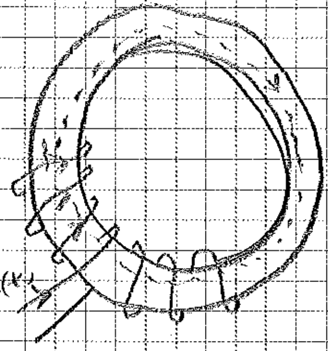


$V_{ab} = L\frac{di}{dt}$, Inductor opposes to the variations in the current flowing through the circuit.

Ex. 30.3:

A toroidal solenoid with cross-sectional area A and mean radius r is closely wound with N turns.

What is L?



Use $L = \dfrac{N\Phi_B}{i}$

$\vec{B}$ is along the tangential direction with constant magnitude.

Remember: $\oint \vec{B}\cdot d\vec{l} = B2\pi r = \mu_0 N i \Rightarrow B(r) = \dfrac{\mu_0 N i}{2\pi r}$

$$\Rightarrow \Phi_B = \int \vec{B}(r)\cdot d\vec{A} \approx A\,\frac{\mu_0 N i}{2\pi r}$$

Φ_B ← flux through one winding

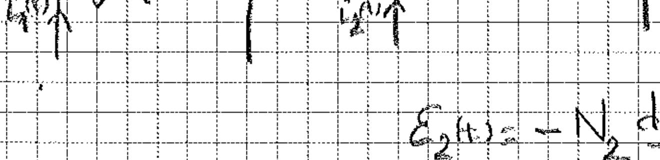
r ← mean radius

$$\Rightarrow L = \frac{N\Phi_B}{i} = \frac{N}{\cancel{i}}\,\frac{A\mu_0 N\cancel{i}}{2\pi r} = \frac{\mu_0 N^2 A}{2\pi r}$$

Mutual Inductance:

A changing current in one coil can also induce an emf in another coil.

Consider two coils:



$$\mathcal{E}_2(t) = -N_2\frac{d\Phi_B(t)}{dt}$$

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We define the mutual inductance;

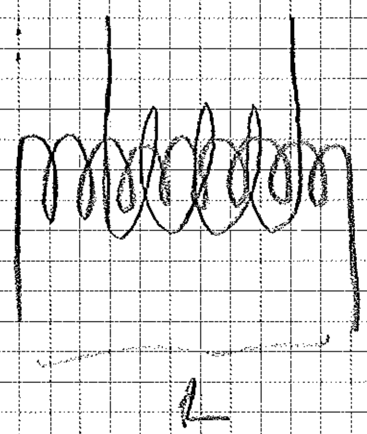
$$M_{21} = \frac{N_2 \Phi_2}{i_1} \quad \text{, mutual inductance}$$

compare with $\left(L = \frac{N\Phi}{i} \right)$

$$\Rightarrow \boxed{\mathcal{E}_2(t) = -M_{21} \frac{di_1}{dt}}$$

induced emf in coil 2 due to changing current in coil 1.

Ex 30.1:



A long solenoid with length L and cross-sectional area A is closely wound with N_1 turns. A coil with N_2 turns surrounds the first coil.

What is the mutual inductance?

$$M_{21} = \frac{N_2 \Phi_2}{i_1}$$

$$\Phi_2 = BA \quad , \quad B = \mu_0 i_1 \frac{N_1}{L}$$

$$= \mu_0 i_1 \frac{N_1}{L} A$$

$$\Rightarrow \boxed{M_{21} = \mu_0 \frac{N_1 N_2}{L} A}$$

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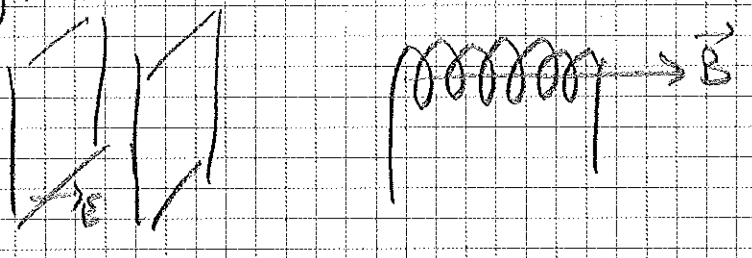
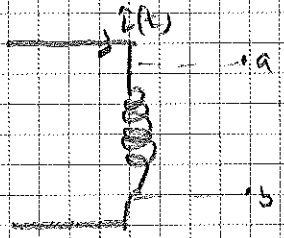
30.3 Magnetic Field Energy:

An inductor carrying current has energy stored in it.
In a capacitor an electric field is built up due to the stored energy.



In an inductor magnetic field is built up due to the stored energy.
Consider an inductor in a circuit:



$$V_{ab} = L\frac{di}{dt}$$

Power stored in the inductor

$\hookrightarrow P = V_{ab}\, i = L\, i \frac{di}{dt} = \frac{dU}{dt}$

$$\Rightarrow \int_0^U dU = \int_0^i L\, i\, di \Rightarrow \boxed{U = \frac{1}{2} L i^2}$$ Energy stored in an inductor.

⇒ When current increases from 0 to I energy is stored in the inductor.
When current decreases from I to 0 inductor acts as a source supplying total energy of $\frac{1}{2} L i^2$
When a steady current flows through an inductor, no energy flows into or out from the inductor.

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Energy stored in the inductor can be viewed as stored in the magnetic field.

⇒ Consider the ideal toroidal solenoid:

$$L = \frac{\mu_0 N^2 A}{2\pi r}$$

$$\Rightarrow U = \frac{1}{2} L i^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{2\pi r} i^2$$

This energy is localised in a volume of $2\pi r A$

⇒ energy density $u = \frac{U}{2\pi r A} = \frac{1}{2} \mu_0 \frac{N^2 i^2}{(2\pi r)^2}$

$B = \frac{\mu_0 N i}{2\pi r}$ inside the toroid ⇒ $\frac{N^2 i^2}{(2\pi r)^2} = \frac{B^2}{\mu_0^2}$

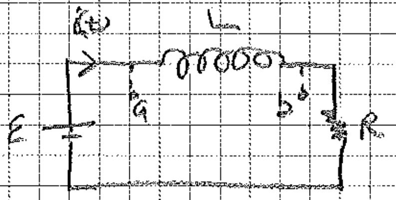
$$\Rightarrow \boxed{u = \frac{B^2}{2\mu_0}}$$ Magnetic energy density in vacuum

This is in general true. For a material with magnetic permeability μ, the energy density is:

$$u = \frac{B^2}{2\mu}$$

30.4 RL Circuit:

A circuit that includes a resistor and an inductor.



What is $i(t) = ?$

$$\mathcal{E} = V_{ab} + V_{bc}$$ "Kirchoff's Loop Rule"

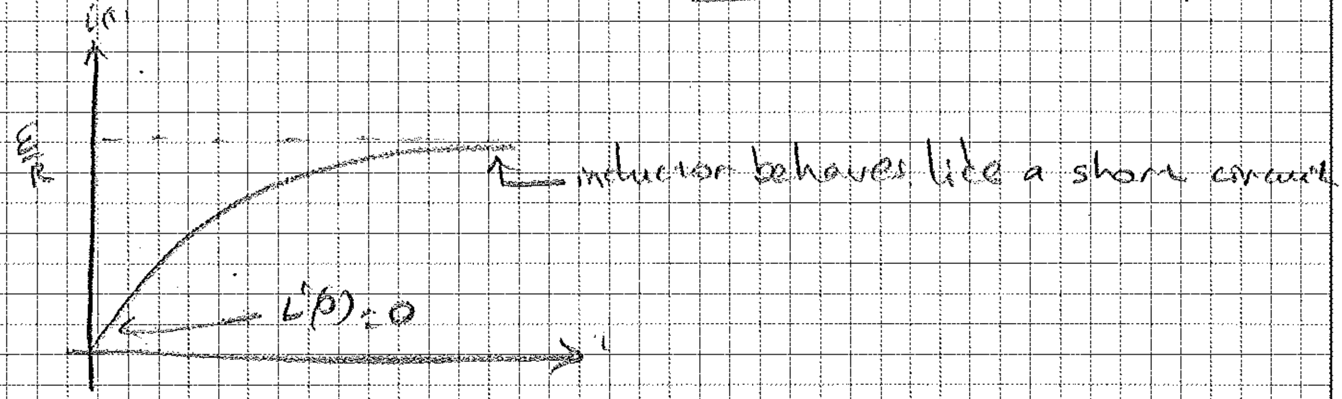
$$\mathcal{E} = L\frac{di}{dt} + iR$$ First order DE.

$$\Rightarrow \frac{\mathcal{E}}{L} - \frac{iR}{L} = \frac{di}{dt}$$

$$\Rightarrow \int \frac{di}{\mathcal{E} - iR} = \int \frac{dt}{L} \Rightarrow -\frac{1}{R}\ln(\mathcal{E} - iR)\Big|_0^{i(t)} = \frac{t}{L}$$

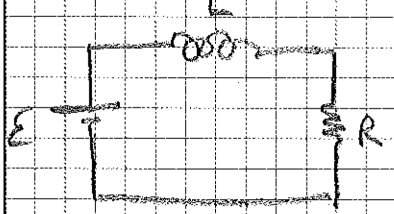
$$\Rightarrow \ln\left(\frac{\mathcal{E} - i(t)R}{\mathcal{E}}\right) = -\frac{R}{L}t$$

$$\Rightarrow 1 - i(t)\frac{R}{\mathcal{E}} = e^{-\frac{R}{L}t} \Rightarrow \boxed{i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$



The time constant is $\boxed{\tau = \frac{L}{R}}$.

Energy wise:



$$i^2R + Li\frac{di}{dt} =$$

$$= \frac{\mathcal{E}^2}{R^2} R\left(1-e^{-\frac{R}{L}t}\right)^2 + \cancel{L}\frac{\mathcal{E}}{R}\left(1-e^{-\frac{R}{L}t}\right)\frac{\mathcal{E}}{R}\frac{R}{\cancel{L}}e^{-\frac{R}{L}t}$$

$$= \frac{\mathcal{E}^2}{R}\left\{1-2e^{-\frac{R}{L}t}+\cancel{e^{-\frac{2R}{L}t}}+e^{-\frac{R}{L}t}-\cancel{e^{-\frac{2R}{L}t}}\right\}$$

$$= \frac{\mathcal{E}^2}{R}\left(1-e^{-\frac{R}{L}t}\right) = \mathcal{E} i$$

$$\Rightarrow \underbrace{\mathcal{E}i}_{\text{power supplied by the battery}} = \underbrace{i^2R}_{\text{power dissipated in R}} + \underbrace{Li\frac{di}{dt}}_{\text{power stored in L}}$$

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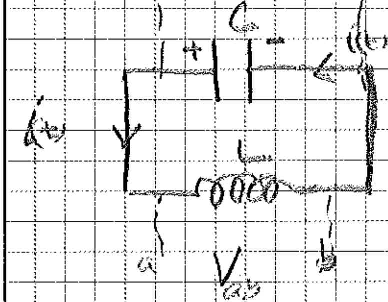
30.5 L - C Circuit:

A circuit containing an ideal inductor and capacitor.



There is no energy dissipating component in the circuit so the energy will be oscillating between the capacitor and inductor.

$$V_C = V_{ab} \Rightarrow \frac{Q}{C} = L\frac{di}{dt}$$

$$i(t) = -\frac{dQ}{dt} \Rightarrow \frac{Q}{C} = -L\frac{d^2Q}{dt^2} \Rightarrow \boxed{\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q}$$

Solution: $Q(t) = A\cos\left(\frac{1}{\sqrt{LC}}t\right) + B\sin\left(\frac{1}{\sqrt{LC}}t\right)$

A, B are constants which depend on the initial conditions. For example if the initial conditions are; $Q(0) = Q_0$, $i(0) = 0$

$$\Rightarrow Q(0) = A = Q_0$$

$$i(0) = -\frac{dQ}{dt}\Big|_{t=0} = -\left\{\frac{1}{\sqrt{LC}}B\right\} = 0 \Rightarrow B = 0$$

$$\Rightarrow \boxed{Q(t) = Q_0\cos\left(\frac{1}{\sqrt{LC}}t\right)}, \quad \omega = \frac{1}{\sqrt{LC}}, \text{ angular frequency of oscillations}$$

Power stored in L; $\frac{1}{2}Li^2 = \frac{1}{2}L\frac{Q_0^2}{LC}\sin^2\left(\frac{1}{\sqrt{LC}}t\right) = \frac{1}{2}\frac{Q_0^2}{C}\sin^2\left(\frac{1}{\sqrt{LC}}t\right)$

Power stored in C: $\frac{1}{2}CQ^2 = \frac{1}{2}\frac{Q_0^2}{C}\cos^2\left(\frac{1}{\sqrt{LC}}t\right)$

$$\Rightarrow \boxed{P_L + P_C = \frac{1}{2}\frac{Q_0^2}{C} = \text{constant}}$$

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Ex 30.9: A 300V dc power is used to charge a $27\mu F$ capacitor. After the capacitor is fully charged, it is disconnected from the power supply, and connected across a 10mH inductor.

a) Frequency and period of oscillations?
b) Capacitor charge and the circuit current 1.2ms after the inductor and capacitor are connected?

a) $\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10\times10^{-3}\times27\times10^{-6}}} = 2.0\times10^{3}\ rad/s$

$\Rightarrow f = \frac{\omega}{2\pi} = 320\,Hz$

$T = \frac{1}{f} = 3.1\,ms$

b) $Q_0 = CV = 27\times10^{-6}\times300 = 81\times10^{-4}\,C$

$\Rightarrow Q(t) = Q_0\cos(\omega t) \Rightarrow Q(1.2ms) = 81\times10^{-4}\cos\left(2.0\times\cancel{10^{3}}\times1.2\times\cancel{10^{-3}}\right)$

$= 81\times10^{-4}\cos(2.4)$

$= -5.5\times10^{-3}\,C$

$i(t) = -\frac{dQ}{dt} = Q_0\,\omega\sin\omega t \Rightarrow i(1.2ms) = 81\times10^{-4}\times2\times10^{3}\sin(2.4)$

$= 10A$ //

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$$Q(t) = e^{-\frac{R}{2L}t}\left\{ A e^{\frac{1}{2}\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}\,t} + B e^{-\frac{1}{2}\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}\,t} \right\}$$

There are 3 regimes of operation depending on the values of R, L, C.

(i) $\frac{R^2}{L^2} = \frac{4}{LC} \Rightarrow Q(t) = A e^{-\frac{R}{2L}t}$, critically damped case

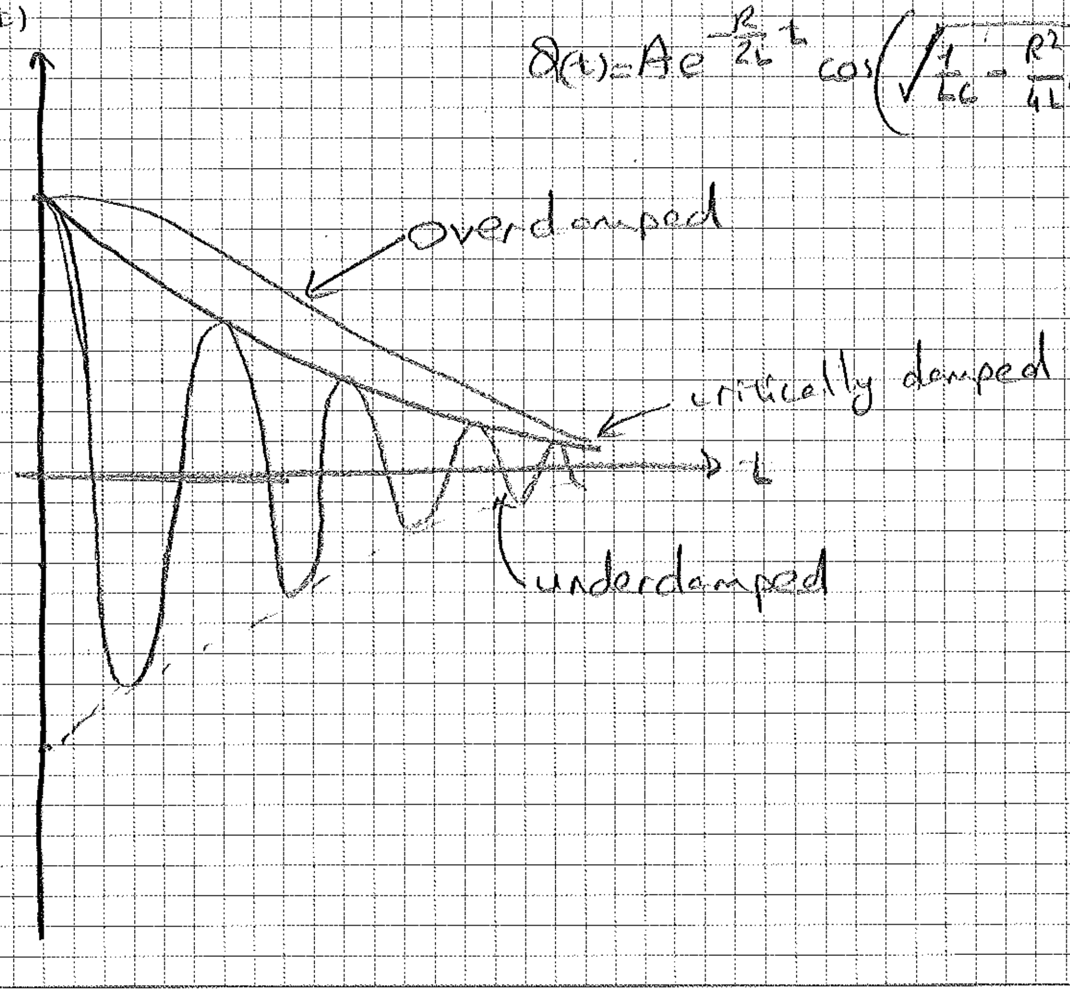
(ii) $\frac{R^2}{L^2} > \frac{4}{LC} \Rightarrow$ overdamped case

(iii) $\frac{R^2}{L^2} < \frac{4}{LC} \Rightarrow$ under damped case

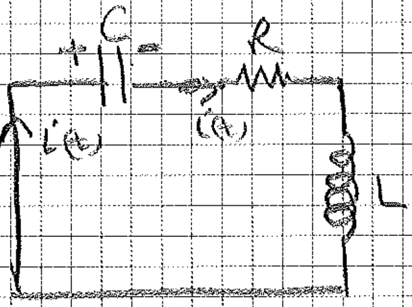
$$Q(t) = e^{-\frac{R}{2L}t}\left\{ A e^{\frac{i}{2}\sqrt{\frac{4}{LC}-\frac{R^2}{L^2}}\,t} + B e^{-\frac{i}{2}\sqrt{\frac{4}{LC}-\frac{R^2}{L^2}}\,t} \right\}$$

oscillations

$$Q(t) = A e^{-\frac{R}{2L}t} \cos\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\;t + \phi\right)$$



30.6. R-L-C Circuit



This time there is an energy dissipating term in the circuit. So, the initial energy stored in the C (or L) will reach to 0 at large times.

$$V_C + iR = -V_L \Rightarrow \frac{Q}{C} + iR = -L\frac{di}{dt}$$

$$i = \frac{dQ}{dt} \Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$$, 2nd order differential equation

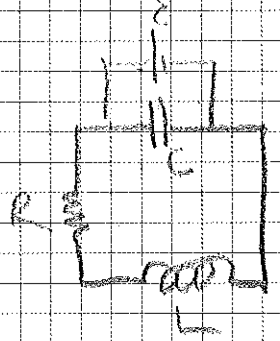
We try solutions of the form:

$$Q(t) = Ae^{kt} \Rightarrow k^2 + \frac{R}{L}k + \frac{1}{LC} = 0 \qquad k_1, k_2$$

$$\Rightarrow k = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

$\Rightarrow$ General solution: $Q(t) = Ae^{k_1 t} + Be^{k_2 t}$

$$Q(t) = e^{-\frac{R}{2L}t}\left\{Ae^{\frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\,t} + Be^{-\frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\,t}\right\}$$



In the circuit we first charge the capacitor using a battery.
Then we disconnect the battery from the circuit.

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Ex 30.11: What is the R value required to give an R-L-C circuit a frequency that is $\frac{1}{2}$ of the undamped freq.?

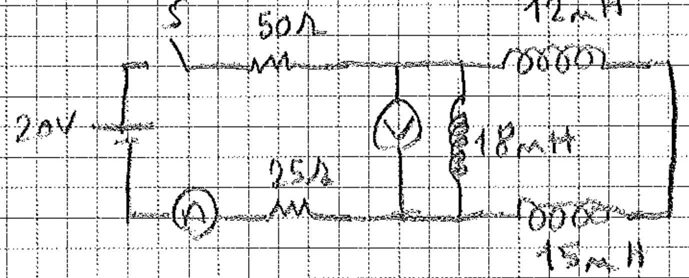
Underdamped frequency: $\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

Undamped frequency, $R=0$: $\sqrt{\frac{1}{LC}}$

$$\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2}\sqrt{\frac{1}{LC}} \Rightarrow \frac{1}{LC} - \frac{R^2}{4L^2} = \frac{1}{4}\frac{1}{LC}$$

$$\Rightarrow \frac{R^2}{4L^2} = \frac{3}{4}\frac{1}{LC} \Rightarrow R^2 = 3\frac{L}{C} \Rightarrow R = \sqrt{\frac{3L}{C}}$$

Prob. 30.73:



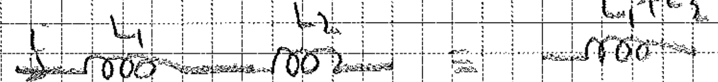
S has been open for a long time. S is suddenly closed.

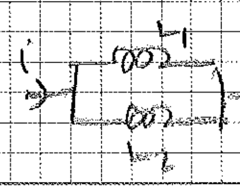
a) What are the readings in A and V just after the switch is closed?
b) What " " " " after a long time?
c) " " " " " at $t = 0.115$ ms?

b) As $t \to \infty$ all inductors behave like short circuits.

$$\Rightarrow V = 0, \quad A = \frac{20V}{75\Omega}$$

a) At $t=0$ $A = 0$, $V = 20V$

c) 



$$V = L_1\frac{di_1}{dt} = L_2\frac{di_2}{dt} = L_{eq}\frac{d(i_1+i_2)}{dt}$$

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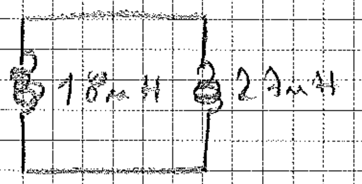
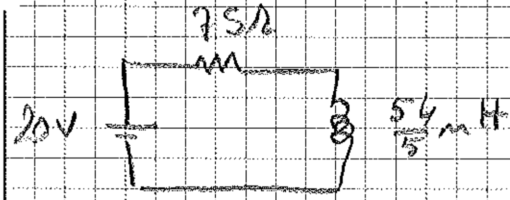
$\Rightarrow L_{eq}\left(\frac{V}{L_1}+\frac{V}{L_2}\right)=V \Rightarrow \frac{1}{L_{eq}}=\frac{1}{L_1}+\frac{1}{L_2}$ //

$\Rightarrow$



$\frac{1}{L_{eq}}=\frac{1}{18}+\frac{1}{27}=\frac{5}{54mH} \Rightarrow L_{eq}=\frac{54}{5}mH$ //

$\Rightarrow$ R-L circuit

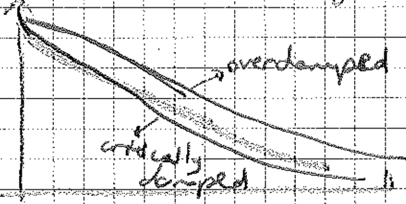


$i(t)=\frac{20}{75}\left(1-e^{-\frac{75}{\frac{54}{5}\times10^{-3}}t}\right)$ //

$V(t)=\left(\frac{54}{5}\times10^{-3}\right)\frac{di}{dt}$ //

When $R^2 = \frac{4L}{C}$, the system no longer oscillates.

↳ critically damped case.



When $R^2 > \frac{4L}{C}$, → overdamped. Capacitor charge approaches zero even more slowly.

Ex: What resistance R is required to give an L-R-C circuit an oscillation frequency that is $\frac{1}{2}$ the undamped frequency.

undamped frequency: $\sqrt{\frac{1}{LC}}$

$$\frac{1}{2}\sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \rightarrow \boxed{R = \frac{3L}{C}}$$

Chap 31: Alternating Current

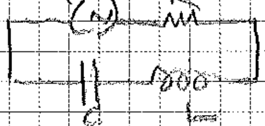
Circuits in which voltages and currents vary sinusoidally are called as alternating current circuits.

Phasors and Alternating Currents

"Most present day household and industrial power-distribution systems operate with alt. current." "In Turkey, 220V"

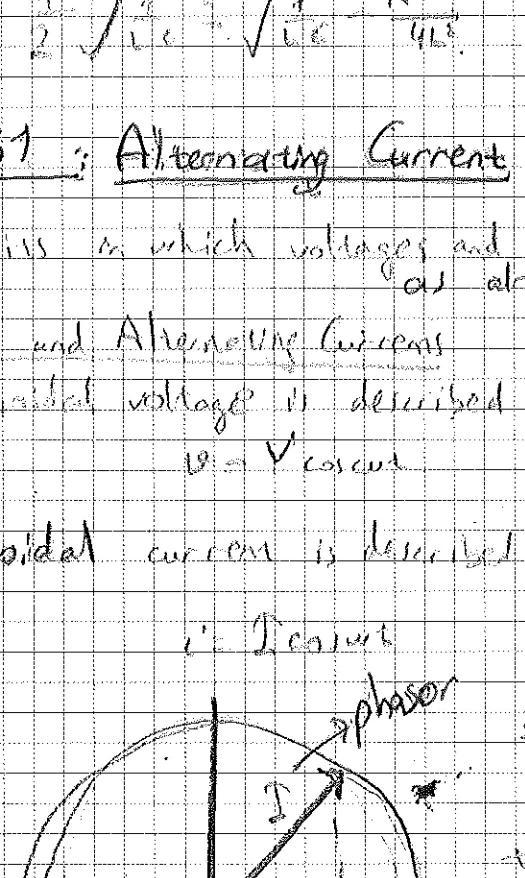
A sinusoidal voltage is described as:

$v = V\cos\omega t$



A sinusoidal current is described as:

$i = I\cos\omega t$



We use phasor diagrams to represent sinusoidally varying voltages and currents.
— The instantaneous value of a quantity that varies sinusoidally with time is represented by the projection onto a horizontal axis of a vector with a length I.

"Phasor is the vector in 2D whose projection to the horizontal axis gives the instantaneous value of the variable."